Consistency of Strictly Impredicative NF

and a little more...

Sergei Tupailo*

Centro de Matemática e Aplicações Fundamentais
Universidade de Lisboa
Avenida Professor Gama Pinto 2
1649-003 Lisboa, Portugal
sergei@cs.ioc.ee

December 21, 2009

Abstract

An instance of Stratified Comprehension

$$\forall x_1 \ldots \forall x_n \exists y \forall x (x \in y \leftrightarrow \phi(x, x_1, \ldots, x_n))$$

is called strictly impredicative iff, under minimal stratification, the type of $x$ is 0. Using the technology of forcing, we prove that the fragment of NF based on strictly impredicative Stratified Comprehension is consistent. A crucial part in this proof, namely showing genericity of a certain symmetric filter, is due to Robert Solovay.

As a bonus, our interpretation also satisfies some instances of Stratified Comprehension which are not strictly impredicative. For example, it verifies existence of Frege natural numbers.

*Acknowledgements: (1) Research partly supported by the Estonian Science Foundation grant ETF7134 and the Institute of Cybernetics at TUT, Tallinn, Estonia; (2) The main body of this work was done at Stanford University, U.S.A., during 2007-08 academic year. The visit was made possible by the Fulbright grant # 07-31122. The result was reported to a conference "NF in the Bay Area", Stanford University, June 25–27, 2008, http://www.cs.ioc.ee/~sergei/nfbayarea08.html; (3) Research partly supported by the U.K. Royal Society International Joint Projects award 2007/R1; (4) Anonymous referee report and suggestions are gratefully acknowledged.
Apparently, this is a new subsystem of \textbf{NF} shown to be consistent. The consistency question for the whole theory \textbf{NF} remains open (since 1937).

\section*{Introduction}

\textit{New Foundations}, \textbf{NF}, is a system of set theory named after Quine’s 1937 article [9] ”New foundations for mathematical logic”, where it was introduced. The language $L_\in$ of \textbf{NF} is the simple set-theoretic language, i.e. the usual first-order language with the only constants $=$ and $\in$. The logic is classical first-order with equality. The only non-logical axioms are \textit{Extensionality} and \textit{Stratified Comprehension} as described below.

\textit{Extensionality} is an axiom
\begin{equation*}
\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).
\end{equation*}

\textbf{Definition 1} Stratification of a formula $\varphi$ is an assignment of natural numbers to variables (both free and bound) in $\varphi$ s.t. every atomic subformula $x = y$ of $\varphi$ receives an assignment $x^n = y^n$, for some $n$, and every atomic subformula $x \in y$ of $\varphi$ receives an assignment $x^m \in y^{m+1}$, for some $m$. A formula $\varphi$ is stratified iff there exists a stratification of $\varphi$.

Equivalently, a formula is stratified iff it can be obtained from a formula of Simple Type Theory by erasing type indices (and renaming variables if necessary).

\textit{Examples}. The formula $x \in y \land y \in z$ is stratified, but the formula $x \in y \land y \in z$ is not.

\textit{Stratified Comprehension} is an axiom scheme
\begin{equation*}
\exists y \forall x (x \in y \leftrightarrow \varphi[x]),
\end{equation*}

for every stratified formula $\varphi$ with $y$ not free in $\varphi$.

It is known that \textbf{NF} is at least as strong as Simple Type Theory with Infinity, but \textbf{NF} is not known to be consistent, relative to any extension of Zermelo-Fränkel Set Theory, which is clearly the main open problem in the area. Despite a number of valiant attempts by a distinguished group of researchers (J. Rosser, E. Specker, R. Jensen, R. Solovay), the question of consistency of \textbf{NF} relative to any \textbf{ZF}-style set theory has eluded a satisfactory solution for more than 70 years. The problem is undoubtedly very difficult.
That said, there is a wealth of subsystems of \( \text{NF} \) which are known to be consistent. Perhaps the most famous of them is \( \text{NFU} \), so called ”\( \text{NF} \) with Urelements”, introduced by Jensen [7, 1969], which results from \( \text{NF} \) by restricting extensionality to non-empty sets. There are also many extensional consistent subsystems of \( \text{NF} \). To start, we mention Hailperin’s [4, 1944] result that \( \text{NF} \) is finitely axiomatizable. Quite a lot of research has been done on whether we can prove consistency when keeping full Extensionality but restricting \( \text{SCA} \) in various ways. The present paper turned out to be a one in this direction.

In our situation we need to mention Crabbé’s [2, 1982] results, who proved consistency of subsystems of \( \text{NF} \) where \( \text{SCA} \) is subjected to certain predicativity\(^1\) restrictions:

**Definition 2 (Crabbé)**  
An instance of Stratified Comprehension

\[
\exists y \forall x \left( x \in y \leftrightarrow \varphi[x] \right), \tag{1}
\]

is predicative iff there is a stratification of (1) s.t. the indices of bound variables in \( \varphi \) are < \text{type}(y), and the indices of free variables in \( \varphi \) are \( \leq \text{type}(y) \).

\( \text{NFP} \) is a subsystem of \( \text{NF} \) where \( \text{SCA} \) is restricted to predicative instances. \( \text{NFI} \) ("mildly impredicative") is an extension of \( \text{NFP} \) which allows bound variables in \( \varphi \) of types \( \leq \text{type}(y) \).

**Theorem 3 ([Crabbé [2]])**  
Both \( \text{NFP} \) and \( \text{NFI} \) are consistent, where in addition

\[
|\text{NFP}| < |\text{EA}|, \\
|\text{PA}_2| \leq |\text{NFI}| < |\text{PA}_3|.
\]

[2] gives two kinds of proofs: model-theoretic (via countably saturated models) and proof-theoretic (via cut-elimination). Holmes [5, 1999] has elaborated on Crabbé’s result, showing that \( \text{NFI} \) has exactly the strength of 2nd order arithmetic \( \text{PA}_2 \).

The results of our paper are, in a sense, complementary to Crabbé’s:

---

\(^1\)We use this terminology here following M. Crabbé. Not to be confused with predicativity in the sense of Feferman, which notion refers to (ordinal) stages of a definition within the same type, not ordering between types as in Crabbé.
Definition 4 An instance of Stratified Comprehension

SCA : \[ \exists y \forall x (x \in y \leftrightarrow \varphi(x)) \],

is strictly impredicative iff there is a stratification of it s.t. the indices of all variables in \( \varphi \) are \( \geq \text{type}(y) - 1 \).

Let NFSI denote a subsystem of NF where SCA is restricted to strictly impredicative instances. Then:

Theorem 5 NFSI (and a little more, e.g. existence of Frege natural numbers) is consistent, too.

The methods we used are set-theoretic (forcing) and entirely different from Crabbé’s.²

The research presented here is motivated by [1] and continues the line of [12].

Before starting, we have to cite one of the backbones of NF-research, due to Specker [10, 1962]. Denoting by TST the Simple Type Theory and by TNT its counterpart where type indices are allowed to run over all (not only non-negative) integers, we have:

Theorem 6 ([Specker [10]])

1. NF is consistent iff there is a model of TNT \( \text{TST is OK} \) with a type-shifting automorphism \( \sigma \).

2. NF is equiconsistent with the Theory of Types, TNTA \( \text{TSTA is OK} \), with the Ambiguity scheme, Amb,

\[ \varphi \leftrightarrow \varphi^+ \],

for all sentences \( \varphi \). \( \varphi^+ \) is the result of raising all type indices in \( \varphi \) by 1.

²Upon circulation of this result M. Crabbé came up with a different (non-forcing) proof of consistency of NFSI. He observed that when \( S \) is any denumerable set and FC\( (X) \) denotes the set of all finite and cofinite subsets of \( X \), then the structure \( S := (S, FC(S), FC(FC(S)), \ldots) \) gives rise to a model of NFSI. Crabbé’s proof of consistency of NFSI alone is simpler (as referee thinks) and uses more elementary means than the one presented here, although the verification that the derived model indeed satisfies NFSI is not entirely trivial. It should be pointed out that Crabbé’s model does not satisfy the "extras" that our model does, e.g. presented in the Sections 2 and 3.
Proof. See [10].

Specker’s proof generalizes immediately to subsystems of NF where SCA is restricted. For NFSI, an equivalent Type Theory is Ext plus Amb plus all instances of
\[ \exists y^{i+1} \forall x^i (x \in y \leftrightarrow \varphi[x]), \]
where all indices in \( \varphi \) are \( \geq i \).

Notations and abbreviations used in the paper. \( \mathbb{P}, \mathbb{P}_i \) and \( G \) will be used for fixed partial orderings and a filter, see the beginning of Section 1. We will use the \texttt{mathbb} font, as \( \mathbb{P} \) and \( G \), to talk about any partial orderings and filters in a given context, see Lemma 9 and further on.

\( f : a \rightarrow b \) says that \( f \) is a function from \( a \) to \( b \), and \( f : a \rightarrow^\text{bi} b \) says that \( f \) is a bijection between \( a \) and \( b \).

\( \text{TST}^n, \text{TSTA}^n \) is a subsystem of \( \text{TST}, \text{TSTA} \), resp., which allows only indices \( i \) satisfying \( 0 \leq i \leq n \).

1 Consistency of NFSI

From the outset, we assume consistency of ZFC. Let \( (M, \in) \) be an Ehrenfeucht-Mostowski model of \( \text{ZF} + \mathbb{V} = \mathbb{L} \), i.e. a countable model with a non-trivial external \( \in \)-automorphism \( \sigma \). Without loss of generality we may assume that \( \sigma \) moves up at least one regular cardinal \( \kappa \) (in the sense of \( M \)):

Proof: In \( M \), sets can be enumerated by ordinals, i.e. there is a formula \( \varphi(x, \alpha) \) s.t. the sentence ”\( \varphi \) gives a (class) bijection between \( \mathbb{V} \) and \( \text{On} \)” is true in \( M \). By Ehrenfeucht-Mostowski, \( \sigma(x) \neq x \) for some \( x \in M \). Since we have a definable bijection, \( \sigma(\alpha) \neq \alpha \) for some ordinal \( \alpha \in M \). If \( \alpha < \sigma(\alpha) \), fine; if not, take \( \sigma^{-1} \).

In order to move up a cardinal, use a definable bijection \( \alpha \mapsto \aleph_\alpha \).

In order to move up a regular cardinal, use a definable injection \( \alpha \mapsto \aleph_{\alpha+1} \). \( \square \)

By default, we will use forcing machinery as laid out in


Although, strictly speaking, we cannot do it, as the exposition in Kunen [8] is for countable \textit{standard transitive} models, and an Ehrenfeucht-Mostowski
model is certainly non-standard, it is well-known that, as far as relative consistency results (which ours are) are concerned, the issue of standardness of a ground model $M$ (w.r.t. the universe $V$), or "physical existence" of a generic filter $G$ and a model $M[G]$, is irrelevant, since forcing can be developed entirely syntactically. We take the freedom of utilizing standard forcing results as presented in Kunen [8] for countable standard models with an understanding that, if necessary, our presentation can be done syntactically without mentioning any models.

Given a finite set $S$ of TSTA-axioms, let $n \geq 2$ be such that all indices $i$ in $S$ fall under $0 \leq i \leq n$. For $0 \leq i < n$, let $\mathbb{P}_i := \text{Fn}(\sigma^i(\kappa), 2, \sigma^i(\kappa))$, where

$$\text{Fn}(\kappa_1, 2, \kappa_0) := \{p \mid |p| < \kappa_0 \land p \text{ is a function} \land \text{dom}(p) \subset \kappa_1 \land \text{ran}(p) \subset 2\}$$

(see VII 6.1), and $\mathbb{P} := \prod_{0 \leq i < n} \mathbb{P}_i$.

Note first that $\sigma$ acts as a bijection between $\sigma^i(\kappa)$ and $\sigma^{i+1}(\kappa)$.

Let $G_0$ be $\mathbb{P}_0$-generic over $M$. Since $\mathbb{P}_0$ is just the poset which makes $\mathcal{P}(\kappa)$ of the size $\sigma(\kappa)$ in a generic extension, we have

$$M[G_0] \models \exists h_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa).$$

The coming Definition 7 and Lemma 8 are not necessary for Consis(NFSI), we could achieve it by working with the original bijection $h_0$ instead of $f_0$ to follow; but choosing a "better" bijection $f_0$ is useful for "bonuses" in Sections 2 and 3.

**Definition 7**

$$\mathcal{P}_{<\omega}(b) := \{a \subset b \mid |a| < \omega\};$$

$$\mathcal{P}_{\geq \omega}(b) := \{a \subset b \mid |a| \geq \omega\}.$$

Let $g_0 \in M$ be such that

$$(g_0 : \kappa \overset{\text{bi}}{\mapsto} \mathcal{P}_{<\omega}(\kappa))^M.$$

Defining $g_i := \sigma^i(g_0)$, we get

$$(g_i : \sigma^i(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}_{<\omega}(\sigma^i(\kappa)))^M.$$

(3)

**Lemma 8** Given $(h_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa))^M[G_0]$ and $(g_0 : \kappa \overset{\text{bi}}{\mapsto} \mathcal{P}_{<\omega}(\kappa))^M$, there exists a bijection $(f_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa))^M[G_0]$ satisfying $(f_0|\kappa = g_0)^M[G_0]$.
Proof. Work in $M[G_0]$. Since $|\mathcal{P}(\kappa)| = \sigma(\kappa)$, $|\mathcal{P}_{<\omega}(\kappa)| = \kappa$ and $\mathcal{P}(\kappa) = \mathcal{P}_{<\omega}(\kappa) \cup \mathcal{P}_{\geq\omega}(\kappa)$, we must have $|\mathcal{P}_{\geq\omega}(\kappa)| = \sigma(\kappa)$, i.e. there is a bijection $h_1$ between $\sigma(\kappa)$ and $\mathcal{P}_{\geq\omega}(\kappa)$. Now, for $a \in \mathcal{P}(\kappa)$, define $f'_0(a)$ by

$$f'_0(a) := \begin{cases} g_0^{-1}(a) & \text{if } a \in \mathcal{P}_{<\omega}(\kappa), \\ \kappa + h^{-1}_1(a) & \text{otherwise.} \end{cases} \quad (4)$$

We claim that $f'_0$ is a special bijection between $\mathcal{P}(\kappa)$ and $\sigma(\kappa)$:

(i) $f'_0(a) < \sigma(\kappa)$ is seen from (4) and the fact that $\sigma(\kappa)$ is an additive principal number, i.e. an ordinal closed under ordinal sum;

(ii) $f'_0$ is onto: if $\alpha < \kappa$, then by the first line of (4) $f'_0(a) = g_0^{-1}(a) = \alpha$ for some $a \in \mathcal{P}_{<\omega}(\kappa)$; otherwise, $\alpha = \kappa + \beta$ for some $\beta < \sigma(\kappa)$, and then $f'_0(a) = \kappa + h^{-1}_1(a)$ for some $a \in \mathcal{P}_{\geq\omega}(\kappa)$;

(iii) $f'_0$ is 1-1 follows from (4) and the fact that both $g_0^{-1}$ and $h^{-1}_1$ are 1-1;

(iv) further, from the first line of (4) we have $f'_0|_{\mathcal{P}_{<\omega}(\kappa)} = g_0^{-1}$.

From (i-iv) above, $f_0$ can be taken as the inverse of $f'_0$.

Choose $(f_0: \sigma(\kappa) \overset{bi}{\rightarrow} \mathcal{P}(\kappa))^{M[G_0]}$ as guaranteed by Lemma 8.

Let $\tau \in M^{P_0}$ be a name for $f_0$, so that

$$M[G_0] \models \tau_{\mathcal{G}_0}: \sigma(\kappa) \overset{bi}{\rightarrow} \mathcal{P}(\kappa). \quad (5)$$

By the Forcing Theorem VII 3.6

$$\exists p \in G_0 \left( p \overset{\ast}{\Rightarrow} \tau: (\sigma(\kappa))^{P_0} \overset{bi}{\rightarrow} \mathcal{P}((\kappa)^{P_0}) \right)^M. \quad (6)$$

Taking $p \in G_0$ from (6) and applying $\sigma^i$ to this formula, we obtain

$$\left( \sigma^i(p) \overset{\ast}{\Rightarrow} \sigma^i(\tau): (\sigma^{i+1}(\kappa))^{P_i} \overset{bi}{\rightarrow} \mathcal{P}((\sigma^i(\kappa))^{P_i}) \right)^M. \quad (7)$$

Define $G_{i+1} := \sigma^i G_i$, $0 \leq i < n - 1$, and $G := \prod_{0 \leq i < n} G_i$. Then each $G_i$ contains $\sigma^i(p)$ and is $P_i$-generic over $M$ – see Lemma 9. It’s easily verified that $G$ is a filter on $P = \prod_{0 \leq i < n} P_i$, but it is less obvious that $G$ is generic (see Theorem 24, not used until proved, the argument is due to R. Solovay). Also observe that $\sigma^i(\tau) \in M^{P_i}$, for each $i$.

**Lemma 9**

$G$ is $P$-generic over $M$ $\iff$ $\sigma^\mathcal{G}$ is $\sigma(P)$-generic over $M$. 

7
Proof. "$G$ is a filter in $\mathbb{P}$" being equivalent to "$\sigma''G$ is a filter in $\sigma(\mathbb{P})$" follows from $\sigma$ being an isomorphism between $\mathbb{P}$ and $\sigma(\mathbb{P})$. For the "generic" part, it follows from "$D$ is dense in $\mathbb{P}$" $\iff$ "$\sigma''D$ is dense in $\sigma(\mathbb{P})$" (an isomorphism) and $\sigma''D = \sigma(D)$ ($\sigma \in \text{autorphism of } M$).

Starting with the complete embeddings $\mathbb{P}_i \rightarrow \prod_{0 \leq i < n} \mathbb{P}_i$, define natural embeddings $\iota_i : M[\mathbb{P}_i] \rightarrow M[\mathbb{P}]$ as in VII 7.12.

Lemma 10 For each $i$, $0 \leq i < n$, $M[G_i]$ is a transitive submodel of $M[G]$.

Proof. Let $x \in M[G_i]$. Then $x = \rho_{G_i}$ for $\rho \in M[\mathbb{P}_i]$. Then $\iota_i(\rho) \in M[\mathbb{P}]$ and $x = \rho_{G_i} = (\iota_i(\rho))_G$ by VII 7.13(a), so that $x \in M[G]$.

Now, assume $x = \rho_{G_i} = (\iota_i(\rho))_G \in M[G_i]$, $y = \tau_G \in M[G]$, $y \in_M[G] x$. We need to show $y \in M[G_i]$ and $y \in_M[G_i] x$. We compute:

\[
y \in M[G_i] x \iff \tau_G \in M[G_i] (\iota_i(\rho))_G \iff \exists p \in G (\langle \tau, p \rangle \in \iota_i(\rho)) \iff \exists p \in G \exists p_i \in G_i \exists \delta \in M[\mathbb{P}_i] (\langle \delta, p_i \rangle \in \rho \land \tau = \iota_i(\delta)) \land p = \langle \emptyset, \ldots, p_i, \ldots, \emptyset \rangle \implies y = \delta_{G_i} \in M[G_i] \land \delta_{G_i} \in M[G_i] \rho_{G_i}.
\]

See Picture 1.

Interpret variables $x^i$ of $L_{\text{TST}}^n$ as $x \in \sigma^i(\kappa)$, and interpret $x^i \in^i y^{i+1}$ as $x \in (\sigma^i(\tau))_{G_i}(y)$. First note that from (7) we have

\[
M[G_i] \models (\sigma^i(\tau))_{G_i} : \sigma^{i+1}(\kappa) \overset{\text{bi}}{\rightarrow} \mathcal{P}(\sigma^i(\kappa)),
\]

for each $0 \leq i < n$. For brevity, we denote

\[
f_i := (\sigma^i(\tau))_{G_i} \in M[G_i] \subset M[G].
\]

From Lemma 8, we have

\[
f_i|\sigma^i(\kappa) = g_i.
\]

Under this interpretation, we want to check all axioms of $\text{TSTA}^n$ in $M[G]$. Since $n \geq 2$ is arbitrary, by Compactness it amounts to checking consistency of the corresponding subsystem of $\text{TSTA}$.

Let’s check Extensionality. First observe
Lemma 11

\[ M[G] \models \text{Extensionality.} \]

**Proof.** Follows from VII 2.14. \hfill \Box

We have to model

\[ \forall x^{i+1}\forall y^{i+1} \left( \forall z^i (z \in^i x \leftrightarrow z \in^i y) \rightarrow x = y \right), \]

i.e. to prove, in \( M[G] \),

\[ \forall x \in^{i+1}(\kappa)\forall y \in^{i+1}(\kappa) \left( \forall z \in^i(\kappa)(z \in f_i(x) \leftrightarrow z \in f_i(y)) \rightarrow x = y \right). \]

(10)

Fix \( x, y \in^{i+1}(\kappa) \), and assume

\[ \forall z \in^i(\kappa)(z \in f_i(x) \leftrightarrow z \in f_i(y)). \]

(11)

Since \((f_i(x), f_i(y)) \in P(\sigma^i(\kappa))^{M[G_i]}\), we have \((f_i(x), f_i(y) \subset \sigma^i(\kappa))^{M[G_i]}\), so by absoluteness \( f_i(x), f_i(y) \subset \sigma^i(\kappa) \). Then, (11) can be reduced to

\[ \forall z \in f_i(x) \leftrightarrow z \in f_i(y), \]

(12)

which implies

\[ f_i(x) = f_i(y) \]

by \textit{Extensionality} of \( M[G] \). Now, since the functions \( f_i \) are 1-1 in \( M[G_i] \) (see (8)), by absoluteness they are 1-1 in \( M[G] \), so we can conclude \( x = y \).

For \textit{Ambiguity}, since \( \sigma \) is a bijection between \( \sigma^i(\kappa) \) and \( \sigma^{i+1}(\kappa) \), it’s enough to model

\[ \forall x^i\forall y^{i+1} \left( x \in^i y \leftrightarrow \sigma(x) \in^{i+1} \sigma(y) \right), \]

i.e. to have, in \( M[G] \),

\[ \forall x \in^i(\kappa)\forall y \in^{i+1}(\kappa) \left( x \in f_i(y) \leftrightarrow \sigma(x) \in f_{i+1}(\sigma(y)) \right); \]

(13)

\[ 0 \leq i < n - 1. \]

Lemma 12 For each \( i, 0 \leq i < n - 1 \), there is an \( \in \)-isomorphism \( \sigma_i \) of \( M[G_i] \) onto \( M[G_{i+1}] \) extending \( \sigma|M \); additionally, \( \sigma_i(f_i) = f_{i+1} \).

**Proof.** See Lemma 13 – Corollary 17. \hfill \Box

9
Coming back to (13), fix \( x \in \sigma^i(\kappa), y \in \sigma^{i+1}(\kappa) \). Assume \( M[G] \models x \in f_i(y) \) (the opposite direction being analogous). Since the formula \( \forall x \in f_i(y) \) is \( \Delta_0 \), by absoluteness
\[
M[G_i] \models x \in f_i(y).
\]
By Lemma 12,
\[
M[G_{i+1}] \models \sigma(x) \in f_{i+1}(\sigma(y)).
\]
By absoluteness again, \( M[G] \models \sigma(x) \in f_{i+1}(\sigma(y)) \).

In the Lemma 13 – Corollary 17 below \( \mathbb{P} \) is one of \( \mathbb{P}_i \)'s, \( 0 \leq i < n - 1 \), and \( G \) is \( \mathbb{P} \)-generic over \( M \).

**Lemma 13** \( \mathbf{\sigma: M^P \leftrightarrow M^{\sigma(P)} \text{ and } \sigma \text{ is an } \varepsilon\text{-isomorphism between } (M^P \times \mathbb{P}, M^P) \text{ and } (M^{\sigma(P)} \times \sigma(\mathbb{P}), M^{\sigma(P)}) \text{ in the sense that for every } \mu, \tau \in M^P \text{ and } p \in \mathbb{P}, \langle \mu, p \rangle \in \tau \iff \langle \sigma(\mu), \sigma(p) \rangle \in \sigma(\tau).} \)

**Proof.** \( \mathbf{\sigma: M^P \leftrightarrow M^{\sigma(P)} \text{ follows from the fact that } \forall \tau \in M^P \text{ is a formula of set theory with parameters } \tau, \mathbb{P}. \varepsilon \text{ is preserved since } \sigma \text{ is an } \varepsilon\text{-automorphism of } M.} \)

**Definition 14**
\[
\Sigma = \{ (\tau_G, (\sigma(\tau))_{\sigma_G}) \mid \tau \in M^P \}.
\]

**Lemma 15** \( \mathbf{\Sigma \text{ is an } \varepsilon\text{-isomorphism between } M[G] \text{ and } M[\sigma''G].} \)

**Proof.** We need to check four things: (a) \( \Sigma \) is a function; (b) \( \Sigma \) is onto; (c) \( \Sigma \) is 1-1; (d) \( \Sigma \) commutes with \( \varepsilon \). (b) follows from the fact that \( \sigma: M^P \rightarrow M^{\sigma(P)} \) is onto, Lemma 13. (d): Let \( \mu_G \in \tau_G \). Then \( \exists \mu \in G \langle \mu, p \rangle \in \tau \). Then \( \sigma(p) \in \sigma''G \) and \( (\sigma(\mu), \sigma(p)) \in \sigma(\tau) \) (Lemma 13). This means \( (\sigma(\mu))_{\sigma''G} \in (\sigma(\tau))_{\sigma''G} \). (a) Let \( \tau_G = \tau'_G \). Then \( \forall \mu \in M^P \langle \mu_G \in \tau_G \iff \mu_G \in \tau'_G \rangle \). By (d) \( \forall \mu \in M^P ((\sigma(\mu))_{\sigma''G} \in (\sigma(\tau))_{\sigma''G} \iff (\sigma(\mu))_{\sigma''G} \in (\sigma(\tau'))_{\sigma''G} \). By Lemma 13 this implies \( \forall \mu \in M^{\sigma(P)} (\mu_{\sigma''G} \in (\sigma(\tau))_{\sigma''G} \iff \mu_{\sigma''G} \in (\sigma(\tau'))_{\sigma''G} \), i.e. \( (\sigma(\tau))_{\sigma''G} = (\sigma(\tau'))_{\sigma''G} \). (c) is analogous to (a): \( \Sigma^{-1} \) is a function. 

**Lemma 16** \( \mathbf{\text{For every } x \in M, \langle x \rangle_{\sigma(\mathbb{P})} = \langle x \rangle_{\sigma(\mathbb{P})} \text{ and } \Sigma(x) = \sigma(x).} \)

**Proof.** Since \( 1_P = 1_{\sigma(\mathbb{P})} = 0 \), \( \langle x \rangle_{\sigma(\mathbb{P})} = \langle x \rangle_{\sigma(\mathbb{P})} \) is proved by induction on \( x \). By Definition 14,
\[
\Sigma(x) = \Sigma((\dot{x})_G)_{\sigma(\mathbb{P})} \overset{\text{Def. 14}}{=} (\sigma(\dot{x}))_{\sigma''G} \overset{\text{\sigma-auto}}{=} ((\sigma(x))^{\dot{i}})^{\dot{i}_G} = \sigma(x).
\]

\( \Box \)
Corollary 17

\[ \Sigma \mathcal{M} = \sigma. \]

This concludes the proof of Lemma 12 and verification of the Ambiguity axiom of TSTA\(^n\).

For Comprehension, we want to model

\[ \forall x_1 \ldots \forall x_k \exists y^{i+1} \forall x^i (x \in^i y \leftrightarrow \varphi(x, x_1, \ldots, x_k)). \]

This means to prove, in \( \mathcal{M}[G] \),

\[ \forall x_1 \in \sigma^{i_1}(\kappa) \ldots \forall x_k \in \sigma^{i_k}(\kappa) \exists y \in \sigma^{i+1}(\kappa) \forall x \in \sigma^i(\kappa) \quad (x \in f_i(y) \leftrightarrow \tilde{\varphi}(x, x_1, \ldots, x_k, \sigma^{i_1}(\kappa), \ldots, \sigma^{i_k}(\kappa), f_{j_1}, \ldots, f_{j_l})), \quad (14) \]

where \( \tilde{\varphi} \) is a translation of \( \varphi \) by the rules above.

In this interpretation, \( \mathcal{M}[G] \) does not satisfy (14) for every \( \varphi \). One trivial result is immediate however from what stands: Consis(\( \text{NF}_2 \)). In that case in (14) \( i = j_1 = \ldots = j_l \), and the set

\[ A := \{ x \in \sigma^i(\kappa) \mid \tilde{\varphi}(x, x_1, \ldots, x_k, \sigma^{i_1}(\kappa), \ldots, \sigma^{i_k}(\kappa), f_{j_1}, \ldots, f_{j_l}) \} \quad (15) \]

is in \( \mathcal{M}[G_i] \) by Separation; thus, also having \( A \subset \sigma^i(\kappa) \), \( y \) can be taken to be \( f_i^{-1}(A) \).

—

In general, everything boils down to showing \( A \in \mathcal{M}[G_i] \), but if \( G \) is not generic, it’s even not clear whether \( A \) exists as a set in \( \mathcal{M}[G] \).

—

Assume in addition that \( G \) is \( \mathbb{P} \)-generic over \( M \). Let an instance

\[ \forall x_1 \ldots \forall x_k \exists y \forall x (x \in y \leftrightarrow \varphi(x, x_1, \ldots, x_k)) \quad (16) \]

of Stratified Comprehension be strictly impredicative. In that case, under our interpretation, (16) is true in \( \mathcal{M}[G] \). Indeed, it’s enough to check only the case \( i = 0 \), for the general case follows then by Ambiguity. If \( i = 0 \), then \( A \in \mathcal{M}[G] \) since \( \mathcal{M}[G] \models \text{Separation} \), and actually \( A \in \mathcal{M}[G_0] \) by VII 6.14 (forcing above doesn’t add subsets of smaller cardinals).

—

We can do more, even without assuming \( G \) being generic or an axiom strictly impredicative. For example, the axioms
P7: \( \forall u \exists v \forall x \forall y \left( \langle y, x \rangle \in u \iff \langle x, y \rangle \in v \right) \)

P8: \( \exists v \forall x \left( x \in v \iff \exists y \left( x = \{ y \} \right) \right) \)

can be pushed through utilizing our careful choice of the initial bijection \( f_0 \) as in Lemma 8. See Sections 2 and 3.\(^3\)

What remains now is to show that \( G = \prod_{0 \leq i < n} G_i \) is generic over \( M \). This argument is due to R. Solovay. It uses the notion of \( \lambda \)-closedness, the Product Lemma, and the fact VII 6.14 (forcing above doesn’t add subsets of smaller cardinals):

**Lemma 18** (see VII Exercise B5) Assume \( A \in M \), \( f : A \rightarrow M \) and \( f \in M[\mathcal{G}] \). Then there is a \( B \in M \) such that \( f : A \rightarrow B \).

**Proof.** Let \( f = \tau_G \). We have
\[
\forall x \in A \exists! b \in M \langle x, b \rangle \in f;
\]
this yields
\[
\forall x \in A \exists! b \in M \left( \langle \text{op}(\check{x}, \check{b}) \rangle_G \in \tau_G \right)^{M[\mathcal{G}]},
\]
By VII 3.6
\[
\forall x \in A \exists! b \in M \exists p \in \mathcal{G} \left( p \models^* \text{op}(\check{x}, \check{b}) \in \tau \right)^M,
\]
implying
\[
\forall x \in A \exists! b \in M \exists p \in \mathcal{P} \left( p \models^* \text{op}(\check{x}, \check{b}) \in \tau \right)^M,
\]
\[\text{i.e.}\]
\[
M \models \forall x \in A \exists! b \exists p \in \mathcal{P} \left( p \models^* \text{op}(\check{x}, \check{b}) \in \tau \right).
\]
By Replacement (in \( M \))
\[
M \models \exists B = \{ b \mid \exists x \in A \exists p \in \mathcal{P} \left( p \models^* \text{op}(\check{x}, \check{b}) \in \tau \right) \}.
\]
We need to show \( f : A \rightarrow B \). Let \( x \in A \) and \( b \in M \) be such that \( \langle x, b \rangle \in f \)^{M[\mathcal{G}]. Then \( \langle \text{op}(\check{x}, \check{b}) \rangle_G \in \tau_G \)^{M[\mathcal{G}], and, by VII 3.6, \( \exists p \in \mathcal{P} \left( p \models^* \text{op}(\check{x}, \check{b}) \in \tau \right)^M \), i.e. \( b \in B \).

\(^3\)The numbers "P7" and "P8" are from Hailperin’s finite axiomatization of NF, see [3, p. 26].
Corollary 19 (see VII Exercise B6) Assume \( \mathbb{P} \in M \) and \( \alpha \) is an ordinal of \( M \). Then (1) \( \Rightarrow \) (2), where
1. Whenever \( B \in M \), \( \alpha B \cap M = \alpha M \cap M[\mathbb{G}] \);
2. \( \alpha M \cap M = \alpha M \cap M[\mathbb{G}] \).

Proof. Assume (1). \( \alpha M \cap M \subset \alpha M \cap M[\mathbb{G}] \) is obvious, so we need to show the converse. If \( f \in \alpha M \cap M[G] \), then by Lemma 18 there is a \( B \in M \) s.t. \( f \in \alpha B \cap M[G] \). By (1) we have \( f \in \alpha B \cap M \), and, by transitivity of \( M \), \( f \in \alpha M \cap M \). \( \square \)

VII 6.12. Definition. A poset \( \mathbb{P} \) is \( \lambda \)-closed iff whenever \( \gamma < \lambda \) and \( \{ p_\xi | \xi < \gamma \} \) is a decreasing sequence of elements of \( \mathbb{P} \) (i.e., \( \xi < \eta \Rightarrow p_\xi \geq p_\eta \)), then
\[ \exists q \in \mathbb{P} \forall \xi < \gamma q \leq p_\xi. \]

Lemma 20 Assume \( \mathbb{P} \) is \( \lambda \)-closed and \( G \) is \( \mathbb{P} \)-generic over \( M \). Assume \( y \in M[G], y \subset M, (|y| < \lambda)^{M[G]} \). Then \( y \in M \).

Proof. We have an \( \alpha < \lambda \) and an \( f : \alpha \rightarrow y \) with \( f \in M[G] \). By VII 6.14, (1) of Corollary 19 is satisfied; consequently, so is (2). \( f \in \alpha M \), so by (2) \( f \in M \), and thus \( y = \text{ran}(f) \in M \). \( \square \)

Lemma 21 Assume \( \mathbb{P} \) is \( \lambda \)-closed and \( G \) is \( \mathbb{P} \)-generic over \( M \). Assume \( \kappa_0 < \kappa_1 \) and \( \kappa_0 \leq \lambda \). Then \( (\text{Fn}(\kappa_1, 2, \kappa_0))^{M[G]} = (\text{Fn}(\kappa_1, 2, \kappa_0))^{M[G]} \).

Proof. \( (\text{Fn}(\kappa_1, 2, \kappa_0))^{M[G]} \subset (\text{Fn}(\kappa_1, 2, \kappa_0))^{M[G]} \) follows by absoluteness and \( M \subset M[G] \), so we need to show the converse. Let \( p \in M[G] \) and \( (p \in \text{Fn}(\kappa_1, 2, \kappa_0))^{M[G]} \), \( \forall z \in p \exists x \in \kappa_1 \exists i \in 2 z = \langle x, i \rangle \), so \( p \subset M \). We have \( (|p| < \kappa_0 \leq \lambda)^{M[G]} \), so by Lemma 20 \( p \in M \). A bijection \( f \in M[G] \) between some \( \alpha < \kappa_0 \) and \( p \) is actually in \( M \) by VII 6.14, so that \( (p \in \text{Fn}(\kappa_1, 2, \kappa_0))^{M[G]} \).

Lemma 22 Assume \( \mathbb{P} \) is \( \lambda \)-closed and \( G \) is \( \mathbb{P} \)-generic over \( M \). Assume \( \kappa_0 < \kappa_1 \), \( \kappa_0 < \lambda \), and \( G_0 \) is \( \text{Fn}(\kappa_1, 2, \kappa_0) \)-generic over \( M[G] \). Then \( G_0 \) is \( \text{Fn}(\kappa_1, 2, \kappa_0) \)-generic over \( M[G] \).

Proof. By VII 6.10, \( \text{Fn}(\kappa_1, 2, \kappa_0) \) has the \( (2^{<\kappa_0})^{+} \)-c.c. In \( M[G] \), \( 2^{<\kappa_0} = \kappa_0 \), since \( M \models \text{GCH} \) and by VII 6.14 \( M[G] \) doesn’t change powersets of
cardinals below $\lambda$. Therefore, in $M[G]$, every $\text{Fn}(\kappa_1, 2, \kappa_0)$-antichain has cardinality $\leq \kappa_0$.

Now let $D$ be a dense subset of $\text{Fn}(\kappa_1, 2, \kappa_0)$ lying in $M[G]$. We must show that $\mathbb{G}_0$ meets $D$. By Zorn’s Lemma (applied in $M[G]$) let $A$ be a maximal antichain consisting of elements of $D$. Then $A$ has cardinality at most $\kappa_0$. By Lemma 20 $A$ lies in $M$. $A$ is clearly a maximal $\text{Fn}(\kappa_1, 2, \kappa_0)$-antichain in the sense of $M$. But $\mathbb{G}_0$ is $M$-generic. So $\mathbb{G}_0$ meets $A$. Hence $\mathbb{G}_0$ meets $D$.

\[\square\]

**Corollary 23** Under the conditions of Lemma 22, $\mathbb{G}_0 \times \mathbb{G}$ is $\text{Fn}(\kappa_1, 2, \kappa_0) \times \mathbb{P}$-generic over $M$.

**Proof.** Use the Product Lemma VIII 1.4. \[\square\]

**Theorem 24** $G = \prod_{0 \leq i < n} G_i$ is $\mathbb{P} = \prod_{0 \leq i < n} \mathbb{P}_i$-generic over $M$.

**Proof.** By backwards induction on $j$ we prove that $\prod_{j \leq i < n} G_i$ is $\prod_{j \leq i < n} \mathbb{P}_i$-generic over $M$. The claim is obvious for $j = n - 1$; so we assume it for $j$, $0 < j < n$, and try to prove it for $j - 1$. (We remind $\mathbb{P}_{j-1} = \text{Fn}(\sigma^j(\kappa), 2, \sigma^{j-1}(\kappa))$.) By Corollary 23, it’s enough to see that $\prod_{j \leq i < n} \mathbb{P}_i$ is $\sigma^j(\kappa)$-closed. Each $\mathbb{P}_i$ is $\sigma^i(\kappa)$-closed, see VII 6.13. It follows that each $\mathbb{P}_i$ is $\sigma^k(\kappa)$-closed if $i \geq k$, see VII 6.12. It follows that $\prod_{j \leq i < n} \mathbb{P}_i$ is $\sigma^j(\kappa)$-closed, see Jech [6, 15.12].

Finally, we arrive at

**Theorem 25** Strictly impredicative $\text{NF}$ is consistent.

**Proof.** Above. \[\square\]

### 2 Bonus P8

P8.$k$ is an axiom

$$\exists v \forall x \ (x \in v \leftrightarrow \exists y_1 \ldots \exists y_k \ (\bigwedge_{1 \leq i, j \leq k; i \neq j} y_i \neq y_j \land x = \{y_1, \ldots, y_k\})). \quad (17)$$

((17) asserts existence of a Frege natural number $k \geq 1$. Note that (17) is predicative and not strictly impredicative. The reasoning below also works for $k = 0$, when we understand $\bigwedge_\emptyset$ as $\top$ and $\bigvee_\emptyset$ as $\bot$.)

14
That means that we must satisfy the following axiom of $\text{TSTA}^n$:

$$\exists v_i^{i+2} \forall x^{i+1} (x \leftrightarrow \exists y^i_1 \ldots \exists y^i_k (\bigwedge_{1 \leq i, j \leq k; i \neq j} y_i \neq y_j \land x = \{y_1, \ldots, y_k\})).$$

(Our formula $\varphi[x^{i+1}]$ in this case is "$\exists y^i_1 \ldots \exists y^i_k (\bigwedge_{1 \leq i, j \leq k; i \neq j} y_i \neq y_j \land x^{i+1} = \{y_1, \ldots, y_k\})$".)

Given that

$$x^{i+1} = \{y^i_1, \ldots, y^i_k\} \iff \bigwedge_{1 \leq i \leq k} y^i_i \in x^{i+1} \land \forall u^i \in x^{i+1} \bigvee_{1 \leq i \leq k} u = y^i_i,$$  \quad (19)

we see that the translation of (19), for $y_1, \ldots, y_k \in \sigma^i(\kappa), x \in \sigma^{i+1}(\kappa)$, is

$$\bigwedge_{1 \leq i \leq k} y_i \in f_i(x) \land \forall u \in \sigma^i(\kappa) (u \in f_i(x) \rightarrow \bigvee_{1 \leq i \leq k} u = y_i),$$

i.e.

$$f_i(x) = \{y_1, \ldots, y_k\},$$  \quad (20)

and in this case

$$\tilde{\varphi}(x, f_i) := \exists y_1 \in \sigma^i(\kappa) \ldots \exists y_k \in \sigma^i(\kappa) (\bigwedge_{1 \leq i, j \leq k; i \neq j} y_i \neq y_j \land f_i(x) = \{y_1, \ldots, y_k\}).$$

According to p. 11, in order to verify P8.k under our interpretation, we must have

$$A8.k := \{x \in \sigma^{i+1}(\kappa) \mid \exists y_1 \in \sigma^i(\kappa) \ldots \exists y_k \in \sigma^i(\kappa) (\bigwedge_{1 \leq i, j \leq k; i \neq j} y_i \neq y_j \land f_i(x) = \{y_1, \ldots, y_k\}) \} \in M[G_{i+1}].$$  \quad (21)

(Note that $A8.k \in M[G_i] \subset M[G]$ automatically, because $f_i \in M[G_i]$ and $M[G_i]$ satisfies Separation, but what we actually need is $A8.k \in M[G_{i+1}]$.) Can we arrange for this?

What helps here is our special choice of $f_i$'s:

**Claim 26** For $y_1, \ldots, y_k \in \sigma^i(\kappa), x \in \sigma^{i+1}(\kappa), f_i(x) = \{y_1, \ldots, y_k\}$ is equivalent to $x \in \sigma^i(\kappa) \land g_i(x) = \{y_1, \ldots, y_k\}$.

**Proof.** $\iff$: Immediate from (9).

$\Rightarrow$: Let $a = \{y_1, \ldots, y_k\} \in \mathcal{P}_{< \omega}(\sigma^i(\kappa))$. From (8), (3) and (9), $f_i^{-1}(b) = g_i^{-1}(b)$ for $b \in \mathcal{P}_{< \omega}(\sigma^i(\kappa))$ ($f_i^{-1}$ enumerates $\mathcal{P}(\sigma^i(\kappa))$ in a special regular
way). From \( f_i(x) = a \) we have \( x = f_i^{-1}(a) = g_i^{-1}(a) \in \sigma^i(\kappa) \) (see (3)), so we must have \( x \in \sigma^i(\kappa) \land g_i(x) = a \).

\[ \square \]

Summarizing (see (19)–(20)), we have proved

**Lemma 27** Under \( y_1, \ldots, y_k \in \sigma^i(\kappa) \), \( x \in \sigma^{i+1}(\kappa) \), \( (x = \{y_1, \ldots, y_k\})^c \) is equivalent to a \( \Delta_0 \) formula with parameters in \( M \).

Coming back to P8.k, the formula \( \varphi_{8.k}[x^{i+1}] \) in this case is

\[ \exists y_1 \ldots \exists y_k \left( \bigwedge_{1 \leq i, j \leq k, i \neq j} y_i \neq y_j \land x^{i+1} = \{y_1, \ldots, y_k\} \right), \]

and we must check that

\[
A_{8.k} := \{ x \in \sigma^{i+1}(\kappa) \mid \exists y_1 \exists y_k \left( y_i \neq y_j \land (x = \{y_1, \ldots, y_k\})^c \right) \}
\]

is in \( M[G_{i+1}] \). Lemma 27 actually gives us more: \( A_{8.k} \in M \subset M[G_{i+1}] \).

3 **Bonus P7**

P7 is an axiom

\[ \forall u \exists v \forall x \forall y \left( (y, x) \in u \iff (x, y) \in v \right). \]

It follows by logic from

\[ \forall u \exists v \exists z \left( z \in v \iff \exists x \exists y \left( z = \langle x, y \rangle \land \exists t \in u^t = \langle y, x \rangle \right) \right). \]

In Type Theory, it is

\[ \forall i^{i+3} \exists i^{i+3} \exists z \exists^2 \left( z \in v \iff \exists x \exists y \left( z = \langle x, y \rangle \land \exists t \in u^t = \langle y, x \rangle \right) \right). \]

Thus, the formula \( \varphi_7[z^{i+2}] \) in this case is

\[ \exists z^{i+2} \exists z^{i+2} (z = \langle x, y \rangle \land \exists t \in u^t = \langle y, x \rangle). \]

Then, under \( u \in \sigma^{i+3}(\kappa) \), \( z \in \sigma^{i+2}(\kappa) \), \( \varphi_7[z] \) is

\[ \exists x \in \sigma^i(\kappa) \exists y \in \sigma^i(\kappa) \left( ((z = \langle x, y \rangle)^c \land \exists t \in \sigma^{i+2}(\kappa) (t \in f_i(u) \land t = \langle y, x \rangle)) \right). \]
We must check
\[ A7 := \{ z \in \sigma^{i+2}(\kappa) \mid \bar{\varphi}_7[z] \} \in M[G_{i+2}] \].

**Lemma 28** Under \( x, y \in \sigma^i(\kappa), z \in \sigma^{i+2}(\kappa), (z = \langle x, y \rangle)^- \) is equivalent to a \( \Delta_0 \) formula with parameters in \( M \).

**Proof.** A type-theory formula \( z^{i+2} = \langle x^i, y^i \rangle \), under \( x, y \in \sigma^i(\kappa), z \in \sigma^{i+2}(\kappa) \), translates into
\[
\exists u \in \sigma^{i+1}(\kappa) \exists v \in \sigma^{i+1}(\kappa) \left( (u = \{x, x\})^- \land (v = \{x, y\})^- \land (z = \{u, v\})^- \right),
\]
i.e.
\[
\exists u \in \sigma^{i+1}(\kappa) \exists v \in \sigma^{i+1}(\kappa) \left( f_i(u) = \{x, x\} \land f_i(v) = \{x, y\} \land f_{i+1}(z) = \{u, v\} \right),
\]
which by (3) and (9) is equivalent to
\[
\exists u \in \sigma^i(\kappa) \exists v \in \sigma^i(\kappa) \left( g_i(u) = \{x, x\} \land g_i(v) = \{x, y\} \land z \in \sigma^{i+1}(\kappa) \land g_{i+1}(z) = \{u, v\} \right).
\]
By Lemma 28 all parameters in \( (z = \langle x, y \rangle)^- \) and \( (t = \langle y, x \rangle)^- \) in (23) may be assumed to be in \( M \subset M[G_{i+2}] \); also we have \( f_{i+2} \in M[G_{i+2}] \), so by *Separation* \( A7 \in M[G_{i+2}] \).

4 Conclusion

**Theorem 29** NFSI + P8 + P7 *is consistent.*

**Proof.** Above. See Theorem 25 and Sections 2 and 3. \( \square \)

 Surely more small additions to NFSI+P8+P7 can be made to hold in \( M[G] \) under the current interpretation, for example more axioms from Hailperin’s list P1–P9 or from other finite axiomatizations of NF. However, we don’t think that this interpretation has any chances to reach full NF. The consistency of NF problem remains open.
References


