Consistency of Strictly Impredicative **NF** *and a little more...*

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Abstract

An instance of Stratified Comprehension

$$\forall x_1 \ldots \forall x_n \exists y \forall x \, (x \in y \leftrightarrow \phi(x, x_1, \ldots, x_n))$$

is called *strictly impredicative* iff, under minimal stratification, the type of x is 0. Using the technology of forcing, we prove that the fragment of **NF** based on strictly impredicative Stratified Comprehension is consistent. A crucial part in this proof, namely showing genericity of a certain symmetric filter, is due to Robert Solovay.

As a bonus, our interpretation also satisfies some instances of Stratified Comprehension which are *not* strictly impredicative. For example, it verifies existence of Frege natural numbers.

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Apparently, this is a new subsystem of **NF** shown to be consistent. The consistency question for the whole theory **NF** remains open (since 1937).

Introduction

New Foundations, **NF**, is a system of set theory named after Quine's 1937 article [9] *"New foundations for mathematical logic"*, where it was introduced. The language $\mathcal{L}_\in$ of **NF** is the simple set-theoretic language, i.e. the usual first-order language with the only constants $=$ and $\in$. The logic is classical first-order with equality. The only non-logical axioms are *Extensionality* and *Stratified Comprehension* as described below.

Extensionality is an axiom

$$\mathbf{Ext}: \qquad \forall x \forall y \left(\forall z (z \in x \leftrightarrow z \in y) \to x = y \right).$$

Definition 1 Stratification *of a formula φ is an assignment of natural numbers to variables (both free and bound) in φ s.t. every atomic subformula $x = y$ of φ receives an assignment $x^n = y^n$, for some n, and every atomic subformula $x \in y$ of φ receives an assignment $x^m \in y^{m+1}$, for some m. A formula φ is* stratified *iff there exists a stratification of φ.*

Equivalently, a formula is stratified *iff it can be obtained from a formula of Simple Type Theory by erasing type indices (and renaming variables if necessary).*

Examples. The formula $x \in y \wedge y \in z$ is stratified, but the formula $x \in y \wedge y \in x$ is not.

Stratified Comprehension is an axiom scheme

$$\mathbf{SCA}: \qquad \exists y \forall x \left(x \in y \leftrightarrow \varphi[x] \right),$$

for every stratified formula φ with y not free in φ.

It is known that **NF** is at least as strong as Simple Type Theory with Infinity, but **NF** is not known to be consistent, relative to any extension of Zermelo-Fränkel Set Theory, which is clearly the main open problem in the area. Despite a number of valiant attempts by a distinguished group of researchers (J. Rosser, E. Specker, R. Jensen, R. Solovay), the question of consistency of **NF** relative to any **ZF**-style set theory has eluded a satisfactory solution for more than 70 years. The problem is undoubtedly very difficult.

That said, there is a wealth of subsystems of **NF** which *are* known to be consistent. Perhaps the most famous of them is **NFU**, so called "**NF** with Urelements", introduced by Jensen [7, *1969*], which results from **NF** by restricting extensionality to non-empty sets. There are also many *extensional* consistent subsystems of **NF**. To start, we mention Hailperin's [4, *1944*] result that **NF** is finitely axiomatizable. Quite a lot of research has been done on whether we can prove consistency when keeping full *Extensionality* but restricting **SCA** in various ways. The present paper turned out to be a one in this direction.

In our situation we need to mention Crabbé's [2, *1982*] results, who proved consistency of subsystems of **NF** where **SCA** is subjected to certain *predicativity*[1] restrictions:

Definition 2 *(Crabbé) An instance of Stratified Comprehension*

$$\mathbf{SCA}: \qquad \exists y \forall x \left(x \in y \leftrightarrow \varphi[x] \right), \tag{1}$$

is predicative *iff there is a stratification of (1) s.t. the indices of bound variables in φ are $<$ $\mathrm{type}(y)$, and the indices of free variables in φ are $\leq \mathrm{type}(y)$.*

NFP is a subsystem of **NF** where **SCA** is restricted to predicative instances. **NFI** ("mildly impredicative") is an extension of **NFP** which allows bound variables in φ of types $\leq \mathrm{type}(y)$.

Theorem 3 *([Crabbé [2]]) Both* **NFP** *and* **NFI** *are consistent, where in addition*

$$|\mathbf{NFP}| < |\mathbf{EA}|,$$

$$|\mathbf{PA_2}| \leq |\mathbf{NFI}| < |\mathbf{PA_3}|.$$

[2] gives two kinds of proofs: *model-theoretic (via countably saturated models)* and *proof-theoretic (via cut-elimination)*. Holmes [5, *1999*] has elaborated on Crabbé's result, showing that **NFI** has exactly the strength of 2nd order arithmetic $\mathbf{PA_2}$.

The results of our paper are, in a sense, complementary to Crabbé's:

[1]We use this terminology here following M. Crabbé. Not to be confused with *predicativity* in the sense of Feferman, which notion refers to (ordinal) stages of a definition *within the same type*, not ordering *between types* as in Crabbé.

Definition 4 *An instance of Stratified Comprehension*

$$\mathbf{SCA}: \qquad \exists y \forall x \left(x \in y \leftrightarrow \varphi[x] \right),$$

is strictly impredicative *iff there is a stratification of it s.t. the indices of all variables in* φ *are* $\geq \mathrm{type}(y) - 1$*.*

Let **NFSI** denote a subsystem of **NF** where **SCA** is restricted to strictly impredicative instances. Then:

Theorem 5 **NFSI** *(and a little more, e.g. existence of Frege natural numbers) is consistent, too.*

The methods we used are *set-theoretic (forcing)* and entirely different from Crabbé's.[2]

The research presented here is motivated by [1] and continues the line of [12].

Before starting, we have to cite one of the backbones of **NF**-research, due to Specker [10, *1962*]. Denoting by **TST** the Simple Type Theory and by **TNT** its counterpart where type indices are allowed to run over all (not only non-negative) integers, we have:

Theorem 6 *([Specker [10]])*

1. **NF** *is consistent iff there is a model of* **TNT** *[***TST** *is OK] with a type-shifting automorphism* σ*.*
2. **NF** *is equiconsistent with the Theory of Types,* **TNTA** *[***TSTA** *is OK], with the* Ambiguity *scheme,* **Amb**,

$$\varphi \leftrightarrow \varphi^{+},$$

for all sentences φ*. [*φ^{+} *is the result of raising all type indices in* φ *by 1.]*

[2]Upon circulation of this result M. Crabbé came up with a different *(non-forcing)* proof of consistency of **NFSI**. He observed that when S is any denumerable set and $FC(X)$ denotes the set of all finite and cofinite subsets of X, then the structure $\mathbf{S} := \langle S, FC(S), FC(FC(S)), \ldots \rangle$ gives rise to a model of **NFSI**. Crabbé's proof of consistency of **NFSI** alone is simpler (as referee thinks) and uses more elementary means than the one presented here, although the verification that the derived model indeed satisfies **NFSI** is not entirely trivial. It should be pointed out that Crabbé's model does not satisfy the "extras" that our model does, e.g. presented in the Sections 2 and 3.

Proof. See [10]. $\square$

Specker's proof generalizes immediately to subsystems of **NF** where **SCA** is restricted. For **NFSI**, an equivalent Type Theory is **Ext** plus **Amb** plus all instances of

$$\exists y^{i+1} \forall x^i \big(x \in y \leftrightarrow \varphi[x]\big) \,,$$

where all indices in φ are $\geq i$.

Notations and abbreviations used in the paper. $\mathbb{P}$, $\mathbb{P}_i$ and G will be used for fixed partial orderings and a filter, see the beginning of Section 1. We will use the `mathbb` font, as $\mathbb{P}$ and $\mathbb{G}$, to talk about *any* partial orderings and filters in a given context, see Lemma 9 and further on.

$f : a \mapsto b$ says that f is a function from a to b, and $f : a \overset{\text{bi}}{\mapsto} b$ says that f is a bijection between a and b.

$\mathbf{TST}^n$, $\mathbf{TSTA}^n$ is a subsystem of **TST**, **TSTA**, resp., which allows only indices i satisfying $0 \leq i \leq n$.

1 Consistency of NFSI

From the outset, we assume consistency of **ZFC**. Let $\langle M, \in \rangle$ be an Ehrenfeucht-Mostowski model of $\mathbf{ZF} + \mathbf{V} = \mathbf{L}$, i.e. a countable model with a non-trivial external $\in$-automorphism σ. Without loss of generality we may assume that σ moves up at least one regular cardinal κ (in the sense of M):

Proof: In M, sets can be enumerated by ordinals, i.e. there is a formula $\varphi(x, \alpha)$ s.t. the sentence "φ gives a (class) bijection between **V** and **On**" is true in M. By Ehrenfeucht-Mostowski, $\sigma(x) \neq x$ for some $x \in M$. Since we have a definable bijection, $\sigma(\alpha) \neq \alpha$ for some ordinal $\alpha \in M$. If $\alpha < \sigma(\alpha)$, fine; if not, take σ^{-1}.
In order to move up a cardinal, use a definable bijection $\alpha \mapsto \aleph_\alpha$.
In order to move up a regular cardinal, use a definable injection $\alpha \mapsto \aleph_{\alpha+1}$.
$\square$

By default, we will use forcing machinery as laid out in

[8] K. Kunen. **Set Theory. An Introduction to Independence Proofs.** Elsevier, 1980.

Although, strictly speaking, we cannot do it, as the exposition in Kunen [8] is for countable *standard transitive* models, and an Ehrenfeucht-Mostowski model is certainly *non-standard*, it is well-known that, as far as relative consistency results (which ours are) are concerned, the issue of *standardness* of a ground model M (w.r.t. the universe $\mathbf{V}$), or *"physical existence"* of a generic filter $\mathbb{G}$ and a model $M[\mathbb{G}]$, is irrelevant, since forcing can be developed entirely syntactically. We take the freedom of utilizing standard forcing results as presented in Kunen [8] for countable standard models with an understanding that, if necessary, our presentation can be done syntactically without mentioning any models.

Given a finite set S of **TSTA**-axioms, let $n \geq 2$ be such that all indices i in S fall under $0 \leq i \leq n$. For $0 \leq i < n$, let $\mathbb{P}_i := \mathrm{Fn}(\sigma^{i+1}(\kappa), 2, \sigma^i(\kappa))$, where

$$\mathrm{Fn}(\kappa_1, 2, \kappa_0) := \{p \mid |p| < \kappa_0 \wedge p \text{ is a function} \wedge \mathrm{dom}(p) \subset \kappa_1 \wedge \mathrm{ran}(p) \subset 2\} \tag{2}$$

(see VII 6.1), and $\mathbb{P} := \mathbb{P}^n := \prod_{0 \leq i < n} \mathbb{P}_i$.

Note first that σ acts as a bijection between $\sigma^i(\kappa)$ and $\sigma^{i+1}(\kappa)$.

Let G_0 be $\mathbb{P}_0$-generic over M. Since $\mathbb{P}_0$ is just the poset which makes $\mathcal{P}(\kappa)$ of the size $\sigma(\kappa)$ in a generic extension, we have

$$M[G_0] \models \exists h_0 \, h_0 : \sigma(\kappa) \overset{\mathrm{bi}}{\mapsto} \mathcal{P}(\kappa).$$

The coming Definition 7 and Lemma 8 are not necessary for Consis(**NFSI**), we could achieve it by working with the original bijection h_0 instead of f_0 to follow; but choosing a "better" bijection f_0 is useful for "bonuses" in Sections 2 and 3.

Definition 7

$$\mathcal{P}_{<\omega}(b) := \{a \subset b \mid |a| < \omega\};$$
$$\mathcal{P}_{\geq\omega}(b) := \{a \subset b \mid |a| \geq \omega\}.$$

Let $g_0 \in M$ be such that

$$(g_0 : \kappa \overset{\mathrm{bi}}{\mapsto} \mathcal{P}_{<\omega}(\kappa))^M.$$

Defining $g_i := \sigma^i(g_0)$, we get

$$(g_i : \sigma^i(\kappa) \overset{\mathrm{bi}}{\mapsto} \mathcal{P}_{<\omega}(\sigma^i(\kappa)))^M. \tag{3}$$

Lemma 8 *Given* $(h_0 : \sigma(\kappa) \overset{\mathrm{bi}}{\mapsto} \mathcal{P}(\kappa))^{M[G_0]}$ *and* $(g_0 : \kappa \overset{\mathrm{bi}}{\mapsto} \mathcal{P}_{<\omega}(\kappa))^M$*, there exists a bijection* $(f_0 : \sigma(\kappa) \overset{\mathrm{bi}}{\mapsto} \mathcal{P}(\kappa))^{M[G_0]}$ *satisfying* $(f_0 \restriction \kappa = g_0)^{M[G_0]}$.

Proof. Work in $M[G_0]$. Since $|\mathcal{P}(\kappa)| = \sigma(\kappa)$, $|\mathcal{P}_{<\omega}(\kappa)| = \kappa$ and $\mathcal{P}(\kappa) = \mathcal{P}_{<\omega}(\kappa) \bigcup \mathcal{P}_{\geq\omega}(\kappa)$, we must have $|\mathcal{P}_{\geq\omega}(\kappa)| = \sigma(\kappa)$, i.e. there is a bijection h_1 between $\sigma(\kappa)$ and $\mathcal{P}_{\geq\omega}(\kappa)$. Now, for $a \in \mathcal{P}(\kappa)$, define $f_0'(a)$ by

$$f_0'(a) := \begin{cases} g_0^{-1}(a) & \text{if } a \in \mathcal{P}_{<\omega}(\kappa), \\ \kappa + h_1^{-1}(a) & \text{otherwise.} \end{cases} \tag{4}$$

We claim that f_0' is a special bijection between $\mathcal{P}(\kappa)$ and $\sigma(\kappa)$:

(i) $f_0'(a) < \sigma(\kappa)$ is seen from (4) and the fact that $\sigma(\kappa)$ is an additive principal number, i.e. an ordinal closed under ordinal sum;

(ii) f_0' *is onto:* if $\alpha < \kappa$, then by the first line of (4) $f_0'(a) = g_0^{-1}(a) = \alpha$ for some $a \in \mathcal{P}_{<\omega}(\kappa)$; otherwise, $\alpha = \kappa + \beta$ for some $\beta < \sigma(\kappa)$, and then $f_0'(a) = \kappa + h_1^{-1}(a)$ for some $a \in \mathcal{P}_{\geq\omega}(\kappa)$;

(iii) f_0' *is 1-1* follows from (4) and the fact that both g_0^{-1} and h_1^{-1} are 1-1;

(iv) *further,* from the first line of (4) we have $f_0' \restriction \mathcal{P}_{<\omega}(\kappa) = g_0^{-1}$.

From (i-iv) above, f_0 can be taken as the inverse of f_0'.

□

Choose $(f_0 : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa))^{M[G_0]}$ as guaranteed by Lemma 8.

Let $\tau \in M^{\mathbb{P}_0}$ be a name for f_0, so that

$$M[G_0] \models \tau_{G_0} : \sigma(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\kappa). \tag{5}$$

By the Forcing Theorem VII 3.6

$$\exists p \in G_0 \left(p \Vdash^*_{\mathbb{P}_0} \tau : (\sigma(\kappa))\check{}_{\mathbb{P}_0} \overset{\text{bi}}{\mapsto} \mathcal{P}((\kappa)\check{}_{\mathbb{P}_0}) \right)^M . \tag{6}$$

Taking $p \in G_0$ from (6) and applying σ^i to this formula, we obtain

$$\left(\sigma^i(p) \Vdash^*_{\mathbb{P}_i} \sigma^i(\tau) : (\sigma^{i+1}(\kappa))\check{}_{\mathbb{P}_i} \overset{\text{bi}}{\mapsto} \mathcal{P}((\sigma^i(\kappa))\check{}_{\mathbb{P}_i}) \right)^M . \tag{7}$$

Define $G_{i+1} := \sigma'' G_i$, $0 \leq i < n-1$, and $G := \prod_{0 \leq i < n} G_i$. Then each G_i contains $\sigma^i(p)$ and is $\mathbb{P}_i$-generic over M – see Lemma 9. It's easily verified that G is a filter on $\mathbb{P} = \prod_{0 \leq i < n} \mathbb{P}_i$, but it is less obvious that G is generic (see Theorem 24, not used until proved, the argument is due to R. Solovay). Also observe that $\sigma^i(\tau) \in M^{\mathbb{P}_i}$, for each i.

Lemma 9

$$\mathbb{G} \textit{ is } \mathbb{P}\textit{-generic over } M \iff \sigma''\mathbb{G} \textit{ is } \sigma(\mathbb{P})\textit{-generic over } M.$$

Proof. "$\mathbb{G}$ is a filter in $\mathbb{P}$" being equivalent to "$\sigma''\mathbb{G}$ is a filter in $\sigma(\mathbb{P})$" follows from σ being an isomorphism between $\mathbb{P}$ and $\sigma(\mathbb{P})$. For the "generic" part, it follows from "D is dense in $\mathbb{P}$" $\Leftrightarrow$ "$\sigma''D$ is dense in $\sigma(\mathbb{P})$" (σ isomorphism) *and* $\sigma''D = \sigma(D)$ (σ $\in$-automorphism of M). $\square$

Starting with the complete embeddings $\mathbb{P}_i \mapsto \prod_{0\leq i<n} \mathbb{P}_i$, define natural embeddings $\imath_i : M^{\mathbb{P}_i} \mapsto M^{\mathbb{P}}$ as in VII 7.12.

Lemma 10 *For each i, $0 \leq i < n$, $M[G_i]$ is a transitive submodel of $M[G]$.*

Proof. Let $x \in M[G_i]$. Then $x = \rho_{G_i}$ for $\rho \in M^{\mathbb{P}_i}$. Then $\imath_i(\rho) \in M^{\mathbb{P}}$ and $x = \rho_{G_i} = (\imath_i(\rho))_G$ by VII 7.13(a), so that $x \in M[G]$.

Now, assume $x = \rho_{G_i} = (\imath_i(\rho))_G \in M[G_i]$, $y = \tau_G \in M[G]$, $y \in_{M[G]} x$. We need to show $y \in M[G_i]$ and $y \in_{M[G_i]} x$. We compute:

$$\begin{aligned} y \in_{M[G]} x \quad &\Longleftrightarrow \quad \tau_G \in_{M[G]} (\imath_i(\rho))_G \\ &\Longleftrightarrow \quad \exists p \in G\, (\langle \tau, p\rangle \in \imath_i(\rho)) \\ &\overset{\text{VII 7.12}}{\Longleftrightarrow} \quad \exists p \in G\, \exists p_i \in G_i\, \exists \delta \in M^{\mathbb{P}_i}\, (\langle \delta, p_i\rangle \in \rho \wedge \tau = \imath_i(\delta) \\ &\qquad\qquad\qquad \wedge p = \langle \emptyset, \ldots, p_i, \ldots, \emptyset\rangle) \\ &\Longrightarrow \quad y = \delta_{G_i} \in M[G_i] \wedge \delta_{G_i} \in_{M[G_i]} \rho_{G_i}. \end{aligned}$$

$\square$

See Picture 1.

Interpret variables x^i of $\mathcal{L}_{\mathbf{TST}^n}$ as $x \in \sigma^i(\kappa)$, and interpret $x^i \in^i y^{i+1}$ as $x \in (\sigma^i(\tau))_{G_i}(y)$. First note that from (7) we have

$$M[G_i] \models (\sigma^i(\tau))_{G_i} : \sigma^{i+1}(\kappa) \overset{\text{bi}}{\mapsto} \mathcal{P}(\sigma^i(\kappa)), \tag{8}$$

for each $0 \leq i < n$. For brevity, we denote

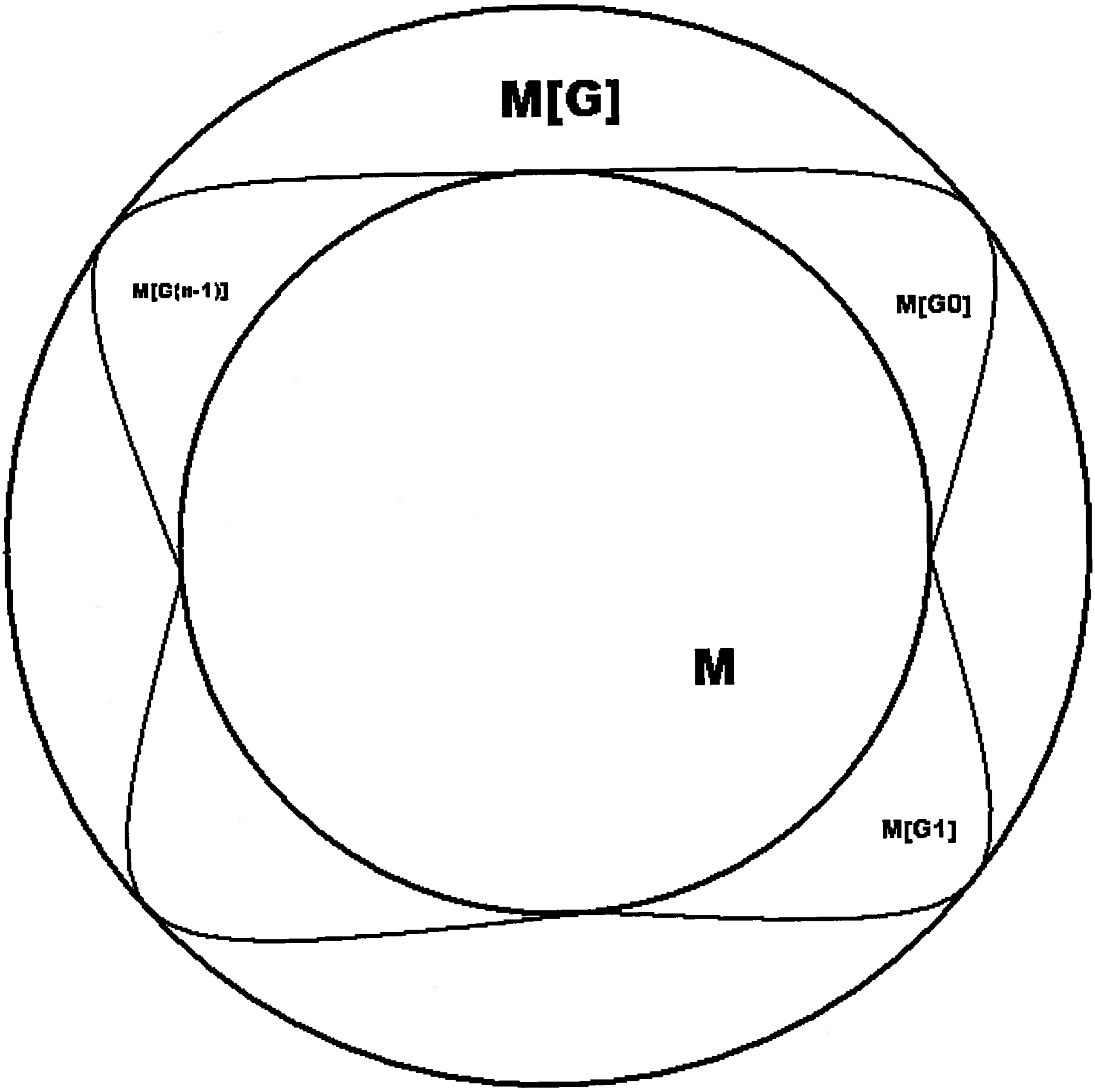
$$f_i := (\sigma^i(\tau))_{G_i} \in M[G_i] \overset{\text{L. 10}}{\subset} M[G].$$

From Lemma 8, we have

$$f_i \restriction \sigma^i(\kappa) = g_i. \tag{9}$$

Under this interpretation, we want to check all axioms of $\mathbf{TSTA}^n$ in $M[G]$. Since $n \geq 2$ is arbitrary, by Compactness it amounts to checking consistency of the corresponding subsystem of $\mathbf{TSTA}$.

Let's check *Extensionality*. First observe



Picture 1

Lemma 11

$$M[G] \models \textit{Extensionality}.$$

Proof. Follows from VII 2.14. □

We have to model

$$\forall x^{i+1} \forall y^{i+1} \left(\forall z^i (z \in^i x \leftrightarrow z \in^i y) \rightarrow x = y \right),$$

i.e. to prove, in $M[G]$,

$$\forall x {\in} \sigma^{i+1}(\kappa) \forall y {\in} \sigma^{i+1}(\kappa) \left(\forall z {\in} \sigma^i(\kappa)(z \in f_i(x) \leftrightarrow z \in f_i(y)) \rightarrow x = y \right). \tag{10}$$

Fix $x, y \in \sigma^{i+1}(\kappa)$, and assume

$$\forall z {\in} \sigma^i(\kappa)(z \in f_i(x) \leftrightarrow z \in f_i(y)). \tag{11}$$

Since $(f_i(x), f_i(y) \in \mathcal{P}(\sigma^i(\kappa)))^{M[G_i]}$, we have $(f_i(x), f_i(y) \subset \sigma^i(\kappa))^{M[G_i]}$, so by absoluteness $f_i(x), f_i(y) \subset \sigma^i(\kappa)$. Then, (11) can be reduced to

$$\forall z \, (z \in f_i(x) \leftrightarrow z \in f_i(y)), \tag{12}$$

which implies

$$f_i(x) = f_i(y)$$

by *Extensionality* of $M[G]$. Now, since the functions f_i are 1-1 in $M[G_i]$ (see (8)), by absoluteness they are 1-1 in $M[G]$, so we can conclude $x = y$.

For *Ambiguity*, since σ is a bijection between $\sigma^i(\kappa)$ and $\sigma^{i+1}(\kappa)$, it's enough to model

$$\forall x^i \forall y^{i+1} \left(x \in^i y \leftrightarrow \sigma(x) \in^{i+1} \sigma(y) \right),$$

i.e. to have, in $M[G]$,

$$\forall x {\in} \sigma^i(\kappa) \forall y {\in} \sigma^{i+1}(\kappa) \left(x \in f_i(y) \leftrightarrow \sigma(x) \in f_{i+1}(\sigma(y)) \right), \tag{13}$$

$0 \leq i < n - 1$.

Lemma 12 *For each i, $0 \leq i < n - 1$, there is an $\in$-isomorphism σ_i of $M[G_i]$ onto $M[G_{i+1}]$ extending $\sigma \restriction M$; additionally, $\sigma_i(f_i) = f_{i+1}$.*

Proof. See Lemma 13 – Corollary 17. □

Coming back to (13), fix $x \in \sigma^i(\kappa)$, $y \in \sigma^{i+1}(\kappa)$. Assume $M[G] \models x \in f_i(y)$ (the opposite direction being analogous). Since the formula "$x \in f_i(y)$" is Δ_0, by absoluteness

$$M[G_i] \models x \in f_i(y).$$

By Lemma 12,

$$M[G_{i+1}] \models \sigma(x) \in f_{i+1}(\sigma(y)).$$

By absoluteness again, $M[G] \models \sigma(x) \in f_{i+1}(\sigma(y))$.

In the Lemma 13 – Corollary 17 below $\mathbb{P}$ is one of $\mathbb{P}_i$'s, $0 \leq i < n-1$, and $\mathbb{G}$ is $\mathbb{P}$-generic over M.

Lemma 13 *$\sigma : M^{\mathbb{P}} \overset{\text{bi}}{\mapsto} M^{\sigma(\mathbb{P})}$ and σ is an $\in$-isomorphism between $\langle M^{\mathbb{P}} \times \mathbb{P}, M^{\mathbb{P}} \rangle$ and $\langle M^{\sigma(\mathbb{P})} \times \sigma(\mathbb{P}), M^{\sigma(\mathbb{P})} \rangle$ in the sense that for every $\mu, \tau \in M^{\mathbb{P}}$ and $p \in \mathbb{P}$,*

$$\langle \mu, p \rangle \in \tau \leftrightarrow \langle \sigma(\mu), \sigma(p) \rangle \in \sigma(\tau).$$

Proof. $\sigma : M^{\mathbb{P}} \overset{\text{bi}}{\mapsto} M^{\sigma(\mathbb{P})}$ follows from the fact that "$\tau \in M^{\mathbb{P}}$" is a formula of set theory with parameters $\tau, \mathbb{P}$. $\in$ is preserved since σ is an $\in$-automorphism of M. □

Definition 14

$$\Sigma \;=\; \{\langle \tau_{\mathbb{G}}, (\sigma(\tau))_{\sigma''\mathbb{G}} \rangle \mid \tau \in M^{\mathbb{P}}\}.$$

Lemma 15 *Σ is an $\in$-isomorphism between $M[\mathbb{G}]$ and $M[\sigma''\mathbb{G}]$.*

Proof. We need to check four things: (a) Σ is a function; (b) Σ is onto; (c) Σ is 1-1; (d) Σ commutes with $\in$. (b) follows from the fact that $\sigma : M^{\mathbb{P}} \mapsto M^{\sigma(\mathbb{P})}$ is onto, Lemma 13. (d): Let $\mu_{\mathbb{G}} \in \tau_{\mathbb{G}}$. Then $\exists p \in \mathbb{G}\, \langle \mu, p \rangle \in \tau$. Then $\sigma(p) \in \sigma''\mathbb{G}$ and $\langle \sigma(\mu), \sigma(p) \rangle \in \sigma(\tau)$ (Lemma 13). This means $(\sigma(\mu))_{\sigma''\mathbb{G}} \in (\sigma(\tau))_{\sigma''\mathbb{G}}$. (a) Let $\tau_{\mathbb{G}} = \tau'_{\mathbb{G}}$. Then $\forall \mu \in M^{\mathbb{P}}\, (\mu_{\mathbb{G}} \in \tau_{\mathbb{G}} \leftrightarrow \mu_{\mathbb{G}} \in \tau'_{\mathbb{G}})$. By (d) $\forall \mu \in M^{\mathbb{P}}\, ((\sigma(\mu))_{\sigma''\mathbb{G}} \in (\sigma(\tau))_{\sigma''\mathbb{G}} \leftrightarrow (\sigma(\mu))_{\sigma''\mathbb{G}} \in (\sigma(\tau'))_{\sigma''\mathbb{G}})$. By Lemma 13 this implies $\forall \mu \in M^{\sigma(\mathbb{P})}\, (\mu_{\sigma''\mathbb{G}} \in (\sigma(\tau))_{\sigma''\mathbb{G}} \leftrightarrow \mu_{\sigma''\mathbb{G}} \in (\sigma(\tau'))_{\sigma''\mathbb{G}})$, i.e. $(\sigma(\tau))_{\sigma''\mathbb{G}} = (\sigma(\tau'))_{\sigma''\mathbb{G}}$. (c) is analogous to (a): Σ^{-1} is a function. □

Lemma 16 *For every $x \in M$, $(x)\check{}_{\mathbb{P}} = (x)\check{}_{\sigma(\mathbb{P})}$ and $\Sigma(x) = \sigma(x)$.*

Proof. Since $1_{\mathbb{P}} = 1_{\sigma(\mathbb{P})} = \emptyset$, $(x)\check{}_{\mathbb{P}} = (x)\check{}_{\sigma(\mathbb{P})}$ is proved by induction on x. By Definition 14,

$$\Sigma(x) = \Sigma((\check{x})_{\mathbb{G}}) \overset{\text{Def. 14}}{=} (\sigma(\check{x}))_{\sigma''\mathbb{G}} \overset{\sigma\ \in\text{-auto}}{=} ((\sigma(x))\check{}\,)_{i''\mathbb{G}} = \sigma(x).$$

□

Corollary 17

$$\Sigma \restriction M = \sigma.$$

This concludes the proof of Lemma 12 and verification of the *Ambiguity* axiom of $\mathbf{TSTA}^n$.

For *Comprehension*, we want to model

$$\forall x_1^{i_1} \dots \forall x_k^{i_k} \exists y^{i+1} \forall x^i \left(x \in^i y \leftrightarrow \varphi(x, x_1, \dots, x_k) \right).$$

This means to prove, in $M[G]$,

$$\begin{aligned} &\forall x_1 \in \sigma^{i_1}(\kappa) \dots \forall x_k \in \sigma^{i_k}(\kappa) \exists y \in \sigma^{i+1}(\kappa) \forall x \in \sigma^i(\kappa) \\ &\quad \left(x \in f_i(y) \leftrightarrow \tilde{\varphi}(x, x_1, \dots, x_k, \sigma^{\iota_1}(\kappa), \dots, \sigma^{\iota_\ell}(\kappa), f_{j_1}, \dots, f_{j_l}) \right), \end{aligned} \tag{14}$$

where $\tilde{\varphi}$ is a translation of φ by the rules above.

In this interpretation, $M[G]$ does not satisfy (14) for every φ. One trivial result is immediate however from what stands: Consis($\mathbf{NF}_2$). In that case in (14) $i = j_1 = \dots = j_l$, and the set

$$A := \{x \in \sigma^i(\kappa) \mid \tilde{\varphi}(x, x_1, \dots, x_k, \sigma^{\iota_1}(\kappa), \dots, \sigma^{\iota_\ell}(\kappa), f_{j_1}, \dots, f_{j_l})\} \tag{15}$$

is in $M[G_i]$ by *Separation*; thus, also having $A \subset \sigma^i(\kappa)$, y can be taken to be $f_i^{-1}(A)$.

—

In general, everything boils down to showing $A \in M[G_i]$, but if G is not generic, it's even not clear whether A exists as a set in $M[G]$.

—

Assume in addition that G is $\mathbb{P}$-generic over M. Let an instance

$$\forall x_1 \dots \forall x_k \exists y \forall x \left(x \in y \leftrightarrow \varphi(x, x_1, \dots, x_k) \right) \tag{16}$$

of *Stratified Comprehension* be *strictly impredicative*. In that case, under our interpretation, (16) is true in $M[G]$. Indeed, it's enough to check only the case $i = 0$, for the general case follows then by *Ambiguity*. If $i = 0$, then $A \in M[G]$ since $M[G] \models$ *Separation*, and actually $A \in M[G_0]$ by VII 6.14 *(forcing above doesn't add subsets of smaller cardinals)*.

—

We can do more, even without assuming G being generic or an axiom strictly impredicative. For example, the axioms

P7: $\forall u \exists v \forall x \forall y\, (\langle y, x\rangle \in u \longleftrightarrow \langle x, y\rangle \in v)$
P8: $\exists v \forall x\, (x \in v \longleftrightarrow \exists y (x = \{y\}))$

can be pushed trough utilizing our careful choice of the initial bijection f_0 as in Lemma 8. See Sections 2 and 3.[3]

—

What remains now is to show that $G = \prod_{0 \le i < n} G_i$ is generic over M. This argument is due to R. Solovay. It uses the notion of λ-closedness, the Product Lemma, and the fact VII 6.14 *(forcing above doesn't add subsets of smaller cardinals)*:

Lemma 18 *(see VII Exercise B5)* *Assume* $A \in M$, $f : A \mapsto M$ *and* $f \in M[\mathbb{G}]$. *Then there is a* $B \in M$ *such that* $f : A \mapsto B$.

Proof. Let $f = \tau_{\mathbb{G}}$. We have

$$\forall x{\in}A\, \exists! b{\in}M\, \langle x, b\rangle \in f;$$

this yields

$$\forall x{\in}A\, \exists! b{\in}M\, ((\mathrm{op}(\check{x}, \check{b}))_{\mathbb{G}} \in \tau_{\mathbb{G}})^{M[\mathbb{G}]}.$$

By VII 3.6

$$\forall x{\in}A\, \exists! b{\in}M\, \exists p{\in}\mathbb{G}\, (p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau)^M,$$

implying

$$\forall x{\in}A\, \exists! b{\in}M\, \exists p{\in}\mathbb{P}\, (p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau)^M,$$

i.e.

$$M \models \forall x{\in}A\, \exists! b\, \exists p{\in}\mathbb{P}\, p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau.$$

By *Replacement* (in M)

$$M \models \exists B = \{b \mid \exists x{\in}A\, \exists p{\in}\mathbb{P}\, p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau\}.$$

We need to show $f : A \mapsto B$. Let $x \in A$ and $b \in M$ be such that $(\langle x, b\rangle \in f)^{M[\mathbb{G}]}$. Then $((\mathrm{op}(\check{x}, \check{b}))_{\mathbb{G}} \in \tau_{\mathbb{G}})^{M[\mathbb{G}]}$, and, by VII 3.6, $\exists p{\in}\mathbb{P}\, (p \Vdash^* \mathrm{op}(\check{x}, \check{b}) \in \tau)^M$, i.e. $b \in B$.

□

[3] The numbers "P7" and "P8" are from Hailperin's finite axiomatization of **NF**, see [3, p. 26].

Corollary 19 *(see VII Exercise B6)* *Assume* $\mathbb{P} \in M$ *and* α *is an ordinal of* M*. Then* (1) $\Rightarrow$ (2)*, where*

(1) *whenever* $B \in M$, ${}^{\alpha}B \cap M = {}^{\alpha}B \cap M[\mathbb{G}]$*;*

(2) ${}^{\alpha}M \cap M = {}^{\alpha}M \cap M[\mathbb{G}]$*.*

Proof. Assume (1). ${}^{\alpha}M \cap M \subset {}^{\alpha}M \cap M[\mathbb{G}]$ is obvious, so we need to show the converse. If $f \in {}^{\alpha}M \cap M[\mathbb{G}]$, then by Lemma 18 there is a $B \in M$ s.t. $f \in {}^{\alpha}B \cap M[\mathbb{G}]$. By (1) we have $f \in {}^{\alpha}B \cap M$, and, by transitivity of M, $f \in {}^{\alpha}M \cap M$. □

VII 6.12. Definition. *A poset* $\mathbb{P}$ *is* λ-closed *iff whenever* $\gamma < \lambda$ *and* $\{p_\xi \,|\, \xi < \gamma\}$ *is a decreasing sequence of elements of* $\mathbb{P}$ *(i.e.,* $\xi < \eta \rightarrow p_\xi \geq p_\eta$*), then*

$$\exists q {\in} \mathbb{P}\, \forall \xi < \gamma\; q \leq p_\xi.$$

Lemma 20 *Assume* $\mathbb{P}$ *is* λ*-closed and* $\mathbb{G}$ *is* $\mathbb{P}$*-generic over* M*. Assume* $y \in M[\mathbb{G}]$*,* $y \subset M$*,* $(|y| < \lambda)^{M[\mathbb{G}]}$*. Then* $y \in M$*.*

Proof. We have an $\alpha < \lambda$ and an $f : \alpha \overset{\text{bi}}{\mapsto} y$ with $f \in M[\mathbb{G}]$. By VII 6.14, (1) of Corollary 19 is satisfied; consequently, so is (2). $f \in {}^{\alpha}M$, so by (2) $f \in M$, and thus $y = \operatorname{ran}(f) \in M$. □

Lemma 21 *Assume* $\mathbb{P}$ *is* λ*-closed and* $\mathbb{G}$ *is* $\mathbb{P}$*-generic over* M*. Assume* $\kappa_0 < \kappa_1$ *and* $\kappa_0 \leq \lambda$*. Then* $(\mathrm{Fn}(\kappa_1, 2, \kappa_0))^M = (\mathrm{Fn}(\kappa_1, 2, \kappa_0))^{M[\mathbb{G}]}$*.*

Proof. $(\mathrm{Fn}(\kappa_1, 2, \kappa_0))^M \subset (\mathrm{Fn}(\kappa_1, 2, \kappa_0))^{M[\mathbb{G}]}$ follows by absoluteness and $M \subset M[\mathbb{G}]$, so we need to show the converse. Let $p \in M[\mathbb{G}]$ and $(p \in \mathrm{Fn}(\kappa_1, 2, \kappa_0))^{M[\mathbb{G}]}$. $\forall z \in p\, \exists x \in \kappa_1\, \exists i \in 2\; z = \langle x, i \rangle$, so $p \subset M$. We have $(|p| < \kappa_0 \leq \lambda)^{M[\mathbb{G}]}$, so by Lemma 20 $p \in M$. A bijection $f \in M[\mathbb{G}]$ between some $\alpha < \kappa_0$ and p is actually in M by VII 6.14, so that $(p \in \mathrm{Fn}(\kappa_1, 2, \kappa_0))^M$. □

Lemma 22 *Assume* $\mathbb{P}$ *is* λ*-closed and* $\mathbb{G}$ *is* $\mathbb{P}$*-generic over* M*. Assume* $\kappa_0 < \kappa_1$*,* $\kappa_0 < \lambda$*, and* $\mathbb{G}_0$ *is* $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$*-generic over* M*. Then* $\mathbb{G}_0$ *is* $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$*-generic over* $M[\mathbb{G}]$*.*

Proof. By VII 6.10, $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$ has the $(2^{<\kappa_0})^+$-c.c. In $M[\mathbb{G}]$, $2^{<\kappa_0} = \kappa_0$, since $M \models$ **GCH** and by VII 6.14 $M[\mathbb{G}]$ doesn't change powersets of cardinals below λ. Therefore, in $M[\mathbb{G}]$, every $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$-antichain has cardinality $\leq \kappa_0$.

Now let D be a dense subset of $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$ lying in $M[\mathbb{G}]$. We must show that $\mathbb{G}_0$ meets D. By Zorn's Lemma (applied in $M[\mathbb{G}]$) let A be a maximal antichain consisting of elements of D. Then A has cardinality at most κ_0. By Lemma 20 A lies in M. A is clearly a maximal $\mathrm{Fn}(\kappa_1, 2, \kappa_0)$-antichain in the sense of M. But $\mathbb{G}_0$ is M-generic. So $\mathbb{G}_0$ meets A. Hence $\mathbb{G}_0$ meets D.

□

Corollary 23 *Under the conditions of Lemma 22,* $\mathbb{G}_0 \times \mathbb{G}$ *is* $\mathrm{Fn}(\kappa_1, 2, \kappa_0) \times \mathbb{P}$*-generic over* M*.*

Proof. Use the Product Lemma VIII 1.4. □

Theorem 24 $G = \prod_{0 \leq i < n} G_i$ *is* $\mathbb{P} = \prod_{0 \leq i < n} \mathbb{P}_i$*-generic over* M*.*

Proof. By backwards induction on j we prove that $\prod_{j \leq i < n} G_i$ is $\prod_{j \leq i < n} \mathbb{P}_i$-generic over M. The claim is obvious for $j = n - 1$; so we assume it for j, $0 < j < n$, and try to prove it for $j - 1$. (We remind $\mathbb{P}_{j-1} = \mathrm{Fn}(\sigma^j(\kappa), 2, \sigma^{j-1}(\kappa))$.) By Corollary 23, it's enough to see that $\prod_{j \leq i < n} \mathbb{P}_i$ is $\sigma^j(\kappa)$-closed. Each $\mathbb{P}_i$ is $\sigma^i(\kappa)$-closed, see VII 6.13. It follows that each $\mathbb{P}_i$ is $\sigma^k(\kappa)$-closed if $i \geq k$, see VII 6.12. It follows that $\prod_{j \leq i < n} \mathbb{P}_i$ is $\sigma^j(\kappa)$-closed, see Jech [6, 15.12]. □

Finally, we arrive at

Theorem 25 *Strictly impredicative* **NF** *is consistent.*

Proof. Above. □

2 Bonus P8

P8.k is an axiom

$$\exists v \forall x \, (x \in v \longleftrightarrow \exists y_1 \ldots \exists y_k \, (\bigwedge_{1 \leq \imath, \jmath \leq k; \imath \neq \jmath} y_\imath \neq y_\jmath \wedge x = \{y_1, \ldots, y_k\})). \quad (17)$$

((17) asserts existence of a Frege natural number $k \geq 1$. Note that (17) is predicative and not strictly impredicative. The reasoning below also works for $k = 0$, when we understand $\bigwedge_\emptyset$ as $\top$ and $\bigvee_\emptyset$ as $\bot$.)

That means that we must satisfy the following axiom of $\mathbf{TSTA}^n$:

$$\exists v^{i+2} \forall x^{i+1} \, (x \in v \longleftrightarrow \exists y_1^i \ldots \exists y_k^i \, (\bigwedge_{1 \leq \imath, \jmath \leq k; \imath \neq \jmath} y_\imath \neq y_\jmath \wedge x = \{y_1, \ldots, y_k\})). \tag{18}$$

(Our formula $\varphi[x^{i+1}]$ in this case is "$\exists y_1^i \ldots \exists y_k^i \, (\bigwedge_{1 \leq \imath, \jmath \leq k; \imath \neq \jmath} y_\imath \neq y_\jmath \wedge x^{i+1} = \{y_1, \ldots, y_k\})$".)

Given that

$$x^{i+1} = \{y_1^i, \ldots, y_k^i\} \iff \bigwedge_{1 \leq \imath \leq k} y_\imath^i \in x^{i+1} \wedge \forall u^i {\in} x^{i+1} \bigvee_{1 \leq \imath \leq k} u = y_\imath^i, \tag{19}$$

we see that the translation of (19), for $y_1, \ldots, y_k \in \sigma^i(\kappa)$, $x \in \sigma^{i+1}(\kappa)$, is

$$\bigwedge_{1 \leq \imath \leq k} y_\imath \in f_i(x) \wedge \forall u {\in} \sigma^i(\kappa) \, (u \in f_i(x) \rightarrow \bigvee_{1 \leq \imath \leq k} u = y_\imath),$$

i.e.

$$f_i(x) = \{y_1, \ldots, y_k\}, \tag{20}$$

and in this case

$$\tilde{\varphi}(x, f_i) \; := \; \exists y_1 {\in} \sigma^i(\kappa) \ldots \exists y_k {\in} \sigma^i(\kappa) \, (\bigwedge_{1 \leq \imath, \jmath \leq k; \imath \neq \jmath} y_\imath \neq y_\jmath \wedge f_i(x) = \{y_1, \ldots, y_k\}).$$

According to p. 11, in order to verify P8.k under our interpretation, we must have

$$A8.k := \begin{array}{c} \{x \in \sigma^{i+1}(\kappa) \mid \exists y_1 {\in} \sigma^i(\kappa) \ldots \exists y_k {\in} \sigma^i(\kappa) \\ (\bigwedge_{1 \leq \imath, \jmath \leq k; \imath \neq \jmath} y_\imath \neq y_\jmath \wedge f_i(x) = \{y_1, \ldots, y_k\})\} \end{array} \in M[G_{i+1}]. \tag{21}$$

(Note that $A8.k \in M[G_i] \subset M[G]$ automatically, because $f_i \in M[G_i]$ and $M[G_i]$ satisfies *Separation*, but what we actually need is $A8.k \in M[G_{i+1}]$.) Can we arrange for this?

What helps here is our special choice of f_i's:

Claim 26 *For $y_1, \ldots, y_k \in \sigma^i(\kappa)$, $x \in \sigma^{i+1}(\kappa)$, $f_i(x) = \{y_1, \ldots, y_k\}$ is equivalent to $x \in \sigma^i(\kappa) \wedge g_i(x) = \{y_1, \ldots, y_k\}$.*

Proof. $\Leftarrow$: Immediate from (9).

$\Rightarrow$: Let $a = \{y_1, \ldots, y_k\} \in \mathcal{P}_{<\omega}(\sigma^i(\kappa))$. From (8), (3) and (9), $f_i^{-1}(b) = g_i^{-1}(b)$ for $b \in \mathcal{P}_{<\omega}(\sigma^i(\kappa))$ (f_i^{-1} enumerates $\mathcal{P}(\sigma^i(\kappa))$ in a special regular way). From $f_i(x) = a$ we have $x = f_i^{-1}(a) = g_i^{-1}(a) \in \sigma^i(\kappa)$ (see (3)), so we must have $x \in \sigma^i(\kappa) \wedge g_i(x) = a$.

□

Summarizing (see (19)–(20)), we have proved

Lemma 27 *Under $y_1, \ldots, y_k \in \sigma^i(\kappa)$, $x \in \sigma^{i+1}(\kappa)$, $(x = \{y_1, \ldots, y_k\})\tilde{}$ is equivalent to a Δ_0 formula with parameters in M.*

—

Coming back to P8.k, the formula $\varphi_{8.k}[x^{i+1}]$ in this case is "$\exists y_1^i \ldots \exists y_k^i \, (\bigwedge_{1 \le \imath, \jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge x^{i+1} = \{y_1, \ldots, y_k\})$", and we must check that

$$\begin{aligned} A8.k &:= \{x \in \sigma^{i+1}(\kappa) \mid \tilde{\varphi}_{8.k}[x]\} \\ &= \{x \in \sigma^{i+1}(\kappa) \mid \exists y_1 \in \sigma^i(\kappa) \ldots \exists y_k \in \sigma^i(\kappa) \\ &\qquad (\bigwedge_{1 \le \imath, \jmath \le k; \imath \ne \jmath} y_\imath \ne y_\jmath \wedge (x = \{y_1, \ldots, y_k\})\tilde{})\} \end{aligned}$$

is in $M[G_{i+1}]$. Lemma 27 actually gives us more: $A8.k \in M \subset M[G_{i+1}]$.

3 Bonus P7

P7 is an axiom

$$\forall u \exists v \forall x \forall y \, (\langle y, x \rangle \in u \longleftrightarrow \langle x, y \rangle \in v).$$

It follows by logic from

$$\forall u \exists v \forall z \left(z \in v \longleftrightarrow \exists x \exists y \, (z = \langle x, y \rangle \wedge \exists t \in u \, t = \langle y, x \rangle) \right).$$

In Type Theory, it is

$$\forall u^{i+3} \exists v^{i+3} \forall z^{i+2} \left(z \in v \longleftrightarrow \exists x^i \exists y^i \, (z = \langle x, y \rangle \wedge \exists t^{i+2} \in u \, t = \langle y, x \rangle) \right). \tag{22}$$

Thus, the formula $\varphi_7[z^{i+2}]$ in this case is

$$\exists x^i \exists y^i \, (z^{i+2} = \langle x, y \rangle \wedge \exists t^{i+2} \in u^{i+3} \, t = \langle y, x \rangle).$$

Then, under $u \in \sigma^{i+3}(\kappa)$, $z \in \sigma^{i+2}(\kappa)$, $\tilde{\varphi}_7[z]$ is

$$\exists x \in \sigma^i(\kappa) \exists y \in \sigma^i(\kappa) \left((z = \langle x, y \rangle)\tilde{} \wedge \exists t \in \sigma^{i+2}(\kappa) \, (t \in f_{i+2}(u) \wedge (t = \langle y, x \rangle)\tilde{}) \right). \tag{23}$$

We must check

$$A7 := \{z \in \sigma^{i+2}(\kappa) \mid \tilde{\varphi}_7[z]\} \in M[G_{i+2}].$$

Lemma 28 *Under $x, y \in \sigma^i(\kappa)$, $z \in \sigma^{i+2}(\kappa)$, $(z = \langle x, y\rangle)\tilde{}$ is equivalent to a Δ_0 formula with parameters in M.*

Proof. A type-theory formula $z^{i+2} = \langle x^i, y^i\rangle$, under $x, y \in \sigma^i(\kappa)$, $z \in \sigma^{i+2}(\kappa)$, translates into

$$\exists u \in \sigma^{i+1}(\kappa)\, \exists v \in \sigma^{i+1}(\kappa)\, \big((u = \{x, x\})\tilde{} \wedge (v = \{x, y\})\tilde{} \wedge (z = \{u, v\})\tilde{}\,\big)\,,$$

i.e.

$$\exists u \in \sigma^{i+1}(\kappa)\, \exists v \in \sigma^{i+1}(\kappa)\, \big(f_i(u) = \{x, x\} \wedge f_i(v) = \{x, y\} \wedge f_{i+1}(z) = \{u, v\}\big)\,,$$

which by (3) and (9) is equivalent to

$$\exists u \in \sigma^{i}(\kappa)\, \exists v \in \sigma^{i}(\kappa)\, \big(g_i(u) = \{x, x\} \wedge g_i(v) = \{x, y\} \wedge z \in \sigma^{i+1}(\kappa) \wedge g_{i+1}(z) = \{u, v\}\big)\,.$$

□

By Lemma 28 all parameters in $(z = \langle x, y\rangle)\tilde{}$ and $(t = \langle y, x\rangle)\tilde{}$ in (23) may be assumed to be in $M \subset M[G_{i+2}]$; also we have $f_{i+2} \in M[G_{i+2}]$, so by *Separation* $A7 \in M[G_{i+2}]$.

4 Conclusion

Theorem 29 **NFSI** $+$ P8 $+$ P7 *is consistent.*

Proof. Above. See Theorem 25 and Sections 2 and 3. □

Surely more small additions to **NFSI**+P8+P7 can be made to hold in $M[G]$ under the current interpretation, for example more axioms from Hailperin's list P1–P9 or from other finite axiomatizations of **NF**. However, we don't think that ***this*** interpretation has any chances to reach full **NF**. The consistency of **NF** problem remains open.

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