Generating Functions

Chapter Seven

Domino Theory and Change
Basic Maneuvers
Solving Recurrences
Special Generating Functions
Convolutions
Exponential Generating Functions
Dirichlet Generating Functions
1 Convolutions
   - Fibonacci convolution
   - $m$-fold convolution
   - Catalan numbers

2 Exponential generating functions
1 Convolutions
   - Fibonacci convolution
   - $m$-fold convolution
   - Catalan numbers

2 Exponential generating functions
Convolutions

- **Given two sequences:**

  \[ \langle f_0, f_1, f_2, \ldots \rangle = \langle f_n \rangle \text{ and } \langle g_0, g_1, g_2, \ldots \rangle = \langle g_n \rangle \]

  The **convolution** of \( \langle f_n \rangle \) and \( \langle g_n \rangle \) is the sequence

  \[ \langle f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 g_1 + f_2 g_0, \ldots \rangle = \left\langle \sum_k f_k g_{n-k} \right\rangle = \left\langle \sum_{k+\ell=n} f_k g_\ell \right\rangle. \]

- If \( F(z) \) and \( G(z) \) are generating functions on the sequences \( \langle f_n \rangle \) and \( \langle g_n \rangle \), then their convolution has the generating function \( F(z) \cdot G(z) \).

- Three or more sequences can be convolved analogously, for example:

  \[ \langle f_n \rangle \langle g_n \rangle \langle h_n \rangle = \left\langle \sum_{j+k+\ell=n} f_j g_k h_\ell \right\rangle \]

  and the generating function of this three-fold convolution is the product \( F(z) \cdot G(z) \cdot H(z) \).
Convolutions

- Given two sequences:

\[
\langle f_0, f_1, f_2, \ldots \rangle = \langle f_n \rangle \quad \text{and} \quad \langle g_0, g_1, g_2, \ldots \rangle = \langle g_n \rangle
\]

The convolution of \( \langle f_n \rangle \) and \( \langle g_n \rangle \) is the sequence

\[
\langle f_0 g_0, f_0 g_1, f_0 g_2 + f_1 g_1, f_0 g_3 + f_1 g_2 + f_2 g_0, \ldots \rangle = \left\langle \sum_k f_k g_{n-k} \right\rangle = \left\langle \sum_{k+\ell=n} f_k g_{\ell} \right\rangle.
\]

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The convolution of \( \langle f_n \rangle \) and \( \langle g_n \rangle \) is the sequence

\[ \langle f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 g_1 + f_2 g_0, \ldots \rangle = \left\langle \sum_k f_k g_{n-k} \right\rangle = \left\langle \sum_{k+\ell=n} f_k g_\ell \right\rangle. \]

If \( F(z) \) and \( G(z) \) are generating functions on the sequences \( \langle f_n \rangle \) and \( \langle g_n \rangle \), then their convolution has the generating function \( F(z) \cdot G(z) \).

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\[ \langle f_n \rangle \langle g_n \rangle \langle h_n \rangle = \left\langle \sum_{j+k+\ell=n} f_j g_k h_\ell \right\rangle \]

and the generating function of this three-fold convolution is the product \( F(z) \cdot G(z) \cdot H(z) \).
1 Convolutions
   - Fibonacci convolution
   - $m$-fold convolution
   - Catalan numbers

2 Exponential generating functions
Fibonacci convolution

To compute \( \sum_k f_k f_{n-k} \) use Fibonacci generating function (in the form given by Theorem 1 and considering that \( \sum (n+1)z^n = \frac{1}{(1-z)^2} \)):

\[
F^2(z) = \left( \frac{1}{\sqrt{5}} \left( \frac{1}{1-\Phi z} - \frac{1}{1-\hat{\Phi} z} \right) \right)^2
= \frac{1}{5} \left( \frac{1}{(1-\Phi z)^2} - \frac{2}{(1-\Phi z)(1-\hat{\Phi} z)} + \frac{1}{(1-\hat{\Phi} z)^2} \right)
= \frac{1}{5} \sum_{n \geq 0} (n+1)\Phi^n z^n - \frac{2}{5} \sum_{n \geq 0} f_{n+1} z^n + \frac{1}{5} \sum_{n \geq 0} (n+1)\hat{\Phi}^n z^n
= \frac{1}{5} \sum_{n \geq 0} (n+1)(\Phi^n + \hat{\Phi}^n) z^n - \frac{2}{5} \sum_{n \geq 0} f_{n+1} z^n
= \frac{1}{5} \sum_{n \geq 0} (n+1)(2f_{n+1} - f_n) z^n - \frac{2}{5} \sum_{n \geq 0} f_{n+1} z^n
= \frac{1}{5} \sum_{n \geq 0} (2nf_{n+1} - (n+1)f_n) z^n
\]

Hence

\[
\sum_k f_k f_{n-k} = \frac{2nf_{n+1} - (n+1)f_n}{5}
\]
Fibonacci convolution (2)

On the previous slide the following was used:

Property

For any $n \geq 0$: $\Phi^n + \hat{\Phi}^n = 2f_{n+1} - f_n$

Proof

The equalities $\sum_n \Phi^n z^n = \frac{1}{1-\Phi z}$, $\sum_n \hat{\Phi}^n z^n = \frac{1}{1-\hat{\Phi} z}$, and $\Phi + \hat{\Phi} = 1$ are used in the following derivation:

$$\sum_n (\Phi^n + \hat{\Phi}^n) z^n = \frac{1}{1-\Phi z} + \frac{1}{1-\hat{\Phi} z} = \frac{1-\hat{\Phi} z + 1-\Phi z}{(1-\Phi z)(1-\hat{\Phi} z)} =$$

$$= \frac{2-z}{1-z-z^2} = \frac{2}{z} \cdot \frac{z}{1-z-z^2} - \frac{z}{1-z-z^2} =$$

$$= \frac{2}{z} \sum_n f_n z^n - \sum_n f_n z^n = 2 \sum_n f_n z^{n-1} - \sum_n f_n z^n =$$

$$= 2 \sum_n f_{n+1} z^n - \sum_n f_n z^n =$$

$$= \sum_n (2f_{n+1} - f_n) z^n$$

Q.E.D.
On the previous slide the following was used:

**Property**

For any \( n \geq 0 \):

\[ \Phi^n + \Phi^n = 2f_{n+1} - f_n \]

**Proof (alternative)**

We know from Exercise 6.28 that

\[ \Phi^n + \Phi^n = L_n = f_{n+1} + f_{n-1} \]

with the convention \( f_{-1} = 1 \), is the \( n \)th Lucas number, which is the solution to the recurrence:

\[
\begin{align*}
L_0 &= 2; \\
L_1 &= 1; \\
L_n &= L_{n-1} + L_{n-2} & \forall n \geq 2.
\end{align*}
\]

By writing the recurrence relation for Fibonacci numbers in the form \( f_{n-1} = f_{n+1} - f_n \) (which, incidentally, yields \( f_{-1} = 1 \)), we get precisely \( L_n = 2f_{n+1} - f_n \).

Q.E.D.
1. **Convolutions**
   - Fibonacci convolution
   - *m*-fold convolution
   - Catalan numbers

2. **Exponential generating functions**
Spanning trees for fan

Example: the fan of order 5:

Spanning trees:

\[ f_1 = 1 \]  \[ f_2 = 3 \]  \[ f_3 = 8 \]
Spanning trees for fan

Example: the fan of order 5:

Spanning trees:

\[ f_1 = 1 \quad f_2 = 3 \quad f_3 = 8 \]
How many spanning trees can we make?

- We need to connect 0 to each of the four blocks:
  - two ways to join 0 with \( \{9, 10\} \),
  - one way to join 0 with \( \{8\} \),
  - four ways to join 0 with \( \{4, 5, 6, 7, \} \),
  - three ways to join 0 with \( \{1, 2, 3\} \),
- There is altogether \( 2 \cdot 1 \cdot 4 \cdot 3 = 24 \) ways for that.

In general:

\[
s_n = \sum_{m > 0} \left( \sum_{k_1 + k_2 + \cdots + k_m = n} k_1 k_2 \cdots k_m \right)
\]

\( k_1, k_2, \ldots, k_m > 0 \)
Spanning trees for fan (2)

How many spanning trees can we make?

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\]

\(k_1, k_2, \ldots, k_m > 0\)

For example

\[
f_4 = 4 + 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 1 \cdot 1 = 21
\]
How many spanning trees can we make?

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  - one way to join 0 with \{8\},
  - four ways to join 0 with \{4, 5, 6, 7, \},
  - three ways to join 0 with \{1, 2, 3\},
- There is altogether \(2 \cdot 1 \cdot 4 \cdot 3 = 24\) ways for that.

In general:

\[
s_n = \sum_{m>0} \left( \sum_{k_1 + k_2 + \cdots + k_m = n} k_1 k_2 \cdots k_m \right)
\]

This is the sum of \(m\)-fold convolutions of the sequence \(\langle 0, 1, 2, 3, \ldots \rangle\).
Spanning trees for fan (3)

**Generating function for the number of spanning trees:**

- The sequence $\langle 0, 1, 2, 3, \ldots \rangle$ has the generating function

  $$G(z) = \frac{z}{(1-z)^2}.$$

- Hence the generating function for $\langle f_n \rangle$ is

  $$S(z) = G(z) + G^2(z) + G^3(z) + \cdots = \frac{G(z)}{1 - G(z)}$$

  $$= \frac{z}{(1-z)^2 \left(1 - \frac{z}{(1-z)^2}\right)}$$

  $$= \frac{z}{(1-z)^2 - z}$$

  $$= \frac{z}{1 - 3z + z^2}.$$
Spanning trees for fan (3)

Generating function for the number of spanning trees:

- The sequence \( \langle 0, 1, 2, 3, \ldots \rangle \) has the generating function

  \[
  G(z) = \frac{z}{(1 - z)^2}.
  \]

- Hence the generating function for \( \langle f_n \rangle \) is

  \[
  S(z) = G(z) + G^2(z) + G^3(z) + \cdots = \frac{G(z)}{1 - G(z)} = \frac{z}{(1 - z)^2(1 - \frac{z}{(1-z)^2})} = \frac{z}{(1 - z)^2 - z} = \frac{z}{1 - 3z + z^2}.
  \]

Consequently \( s_n = f_{2n} \).
1 Convolutions
   • Fibonacci convolution
   • $m$-fold convolution
   • Catalan numbers

2 Exponential generating functions
Dyck language

**Definition**

The Dyck language is the language consisting of balanced strings of parentheses ']' and '['.

**Another definition**

If \( X = \{ x, \overline{x} \} \) is the alphabet, then the **Dyck language** is the subset \( D \) of words \( u \) of \( X^* \) which satisfy

1. \( u|_x = |u|_{\overline{x}} \), where \( |u|_x \) is the number of letters \( x \) in the word \( u \), and
2. if \( u \) is factored as \( vw \), where \( v \) and \( w \) are words of \( X^* \), then \( |v|_x \geq |v|_{\overline{x}} \).
Let $C_n$ be the number of words in the Dyck language $\mathcal{D}$ having exactly $n$ pairs of parentheses.

If $u = vw$ for $u \in \mathcal{D}$, then the prefix $v \in \mathcal{D}$ iff the suffix $w \in \mathcal{D}$.

Then every word $u \in \mathcal{D}$ of length $\geq 2$ has a unique writing $u = [v]w$ such that $v, w \in \mathcal{D}$ (possibly empty) but $[p] \notin \mathcal{D}$ for every prefix $p$ of $u$ (including $u$ itself).

Hence, for any $n > 0$

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$$

The number series $\langle C_n \rangle$ is called Catalan numbers, from the Belgian mathematician Eugène Catalan. Let us derive the closed formula for $C_n$ in the following slides.
**Catalan numbers**

**Step 1** The recurrent equation of Catalan numbers for all integers

\[ C_n = \sum_k C_k C_{n-1-k} + [n = 0]. \]

**Step 2** Write down \( C(z) = \sum_n C_n z^n : \)

\[
C(z) = \sum_n C_n z^n = \sum_{k,n} C_k C_{n-1-k} z^n + \sum_n [n = 0] z^n \\
= \sum_k C_k z^k \sum_n C_{n-1-k} z^{n-1-k} + 1 \\
= \sum_k C_k z^k \sum_n C_n z^n + 1 \\
= zC^2(z) + 1
\]
Catalan numbers

**Step 1** The recurrent equation of Catalan numbers for all integers

\[ C_n = \sum_k C_k C_{n-1-k} + [n = 0]. \]

**Step 2** Write down \( C(z) = \sum_n C_n z^n \):

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C(z) = \sum_n C_n z^n = \sum_{k,n} C_k C_{n-1-k} z^n + \sum_n [n = 0] z^n
\]

\[
= \sum_k C_k z^k \left( \sum_n C_{n-1-k} z^{n-1-k} + 1 \right)
\]

\[
= \sum_k C_k z^k \sum_n C_n z^n + 1
\]

\[
= zC^2(z) + 1
\]
Step 3  Solving the quadratic equation $zC^2(z) - C(z) + 1 = 0$ for $C(z)$ yields directly:

$$C(z) = \frac{1 \pm \sqrt{1 - 4z}}{2z}.$$  

(Solution with "+" isn't proper as it leads to $C_0 = C(0) = \infty$.)

Step 4  From the binomial theorem we get:

$$\sqrt{1 - 4z} = \sum_{k \geq 0} \binom{1/2}{k}(-4z)^k = 1 + \sum_{k \geq 1} \frac{1}{2k} \binom{-1/2}{k-1}(-4z)^k$$

Using the equality for binomials $\binom{-1/2}{n} = (-1/4)^n \binom{2n}{n}$ we finally get

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{k \geq 1} \frac{1}{k} \binom{-1/2}{k-1}(-4z)^{k-1}$$

$$= \sum_{n \geq 0} \binom{-1/2}{n} \frac{(-4z)^n}{n+1}$$

$$= \sum_{n \geq 0} \binom{2n}{n} \frac{z^n}{n+1}$$
Catalan numbers (2)

Step 3 Solving the quadratic equation $zC^2(z) - C(z) + 1 = 0$ for $C(z)$ yields directly:

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$$= \sum_{n \geq 0} \binom{2n}{n} \frac{z^n}{n+1}$$
Proof that \((-\frac{1}{2})_n = (-1/4)^n \binom{2n}{n}\)

We prove a bit more: for every \(r \in \mathbb{R}\) and \(k \geq 0\),

\[ r^k \cdot \left( r - \frac{1}{2} \right)^k = \frac{(2r)^{2k}}{2^{2k}} \]
Proof that \((-\frac{1}{2})^n = (-\frac{1}{4})^n \binom{2n}{n}\)

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r^k \cdot \left( r - \frac{1}{2} \right)^k = \frac{(2r)^{2k}}{2^{2k}}
\]

Indeed,

\[
r^k \cdot \left( r - \frac{1}{2} \right)^k = r \cdot \left( r - \frac{1}{2} \right) \cdot (r - 1) \cdot \left( r - \frac{3}{2} \right) \cdots (r - k - 1) \cdot \left( r - \frac{1}{2} - k + 1 \right)
\]

\[
= \frac{2r}{2} \cdot \frac{2r - 1}{2} \cdot \frac{2r - 2}{2} \cdot \frac{2r - 3}{2} \cdots \frac{2r - 2k - 2}{2} \cdot \frac{2r - 2k + 1}{2}
\]

\[
= \frac{(2r)^{2k}}{2^{2k}}
\]
Proof that \((-1/2)^n = (-1/4)^n (2n\choose n)\)

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Indeed,

\[
\begin{align*}
\left( r - \frac{1}{2} \right)^k &= r \cdot \left( r - \frac{1}{2} \right) \cdot (r-1) \cdot \left( r - \frac{3}{2} \right) \cdots (r-k-1) \cdot \left( r - \frac{1}{2} - k + 1 \right) \\
&= \frac{2r}{2} \cdot \frac{2r-1}{2} \cdot \frac{2r-2}{2} \cdot \frac{2r-3}{2} \cdots \frac{2r-2k-2}{2} \cdot \frac{2r-2k+1}{2} \\
&= \frac{(2r)^{2k}}{2^{2k}}
\end{align*}
\]

Then for \(r = k = n\), dividing by \((n!)^2\) and using \(n^n = n!\),

\[
\left( n - \frac{1}{2} \right) = \left( \frac{1}{4} \right)^n \binom{2n}{n}:
\]

and as \(r^k = (-1)^k (-r)^{\overline{k}} = (-1)^k (-r + k - 1)^{\overline{k}}\),

\[
\binom{-1/2}{n} = \binom{n - (n-1/2) - 1}{n} = \frac{(-1)^n}{4^n} \binom{2n}{n}
\]

Q.E.D.
Resume Catalan numbers

Formulae for computation

- $C_{n+1} = \frac{2(2n+1)}{n+2} C_n$, with $C_0 = 1$
- $C_n = \frac{1}{n+1} \binom{2n}{n}$
- $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \binom{2n-1}{n} - \binom{2n-1}{n+1}$
- Generating function: $C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$

$$\lim_{n \to \infty} \frac{C_n}{C_{n-1}} = 4$$

<table>
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<th>0</th>
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<th>2</th>
<th>3</th>
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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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</thead>
<tbody>
<tr>
<td>$C_n$</td>
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<td>2</td>
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<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
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<td>4 862</td>
<td>16 796</td>
</tr>
</tbody>
</table>
Applications of Catalan numbers

Number of complete binary trees with $n + 1$ leaves is $C_n$

The Dyck language consists of exactly $n$ characters A and $n$ characters B, and every prefix does not contain more B-s than A-s. For example, there are five words with 6 letters in the Dyck language:

AAABBB  AABABB  AABBAB  ABAABB  ABABAB

Corollary
$C_n$ is the number of words of length $2n$ in the Dyck language.
Applications of Catalan numbers

Number of complete binary trees with $n + 1$ leaves is $C_n$

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Corollary

$C_n$ is the number of words of length $2n$ in the Dyck language.
Applications of Catalan numbers (2)

$C_n$ is the number of monotonic paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards.
Applications of Catalan numbers (3)

Polygon triangulation

$C_n$ is the number of different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.

See more applications, for example, on http://www.absoluteastronomy.com/topics/Catalan_number
1 Convolutions
   - Fibonacci convolution
   - $m$-fold convolution
   - Catalan numbers

2 Exponential generating functions
Exponential generating function

Definition

The exponential generating function (briefly, egf) of the sequence \( \langle g_n \rangle \) is the function

\[
\hat{G}(z) = \sum_{n \geq 0} \frac{g_n}{n!} z^n,
\]

that is, the generating function of the sequence \( \langle g_n/n! \rangle \).

For example, \( e^z = \sum_{n \geq 0} \frac{z^n}{n!} \) is the egf of the constant sequence 1.
**Definition**

The **exponential generating function** (briefly, egf) of the sequence \( \langle g_n \rangle \) is the function

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\hat{G}(z) = \sum_{n \geq 0} \frac{g_n}{n!} z^n,
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that is, the generating function of the sequence \( \langle g_n/n! \rangle \).

For example, \( e^z = \sum_{n \geq 0} \frac{z^n}{n!} \) is the egf of the constant sequence 1.

**Why exponential generating functions?**

Because \( \langle g_n/n! \rangle \) might have a “simpler” generating function than \( \langle g_n \rangle \) has.
Let $\hat{F}(z)$ and $\hat{G}(z)$ be the exponential generating functions of $\langle f_n \rangle$ and $\langle g_n \rangle$.

As usual, we put $f_n = g_n = 0$ for every $n < 0$, and undefined $\cdot 0 = 0$.

- $\alpha \hat{F}(z) + \beta \hat{G}(z) = \sum_n \left( \frac{\alpha f_n + \beta g_n}{n!} \right) z^n$
- $\hat{G}(cz) = \sum_n \frac{c^n}{n!} g_n z^n$
- $z \hat{G}(z) = \sum_n \frac{n g_{n-1}}{n!} z^n$
- $\hat{G}'(z) = \sum_n \frac{g_{n+1}}{n!} z^n$
- $\int_0^z \hat{G}(w) dw = \sum_n \frac{g_{n-1}}{n!} z^n$
- $\hat{F}(z) \cdot \hat{G}(z) = \sum_n \frac{1}{n!} \left( \sum_k \binom{n}{k} f_k g_{n-k} \right) z^n$
Let $\hat{F}(z)$ and $\hat{G}(z)$ be the exponential generating functions of $\langle f_n \rangle$ and $\langle g_n \rangle$.

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\[ \alpha \hat{F}(z) + \beta \hat{G}(z) = \sum_n \left( \frac{\alpha f_n + \beta g_n}{n!} \right) z^n \]

\[ \hat{G}(cz) = \sum_n \frac{c^n g_n}{n!} z^n \]

\[ z \hat{G}(z) = \sum_n \frac{n g_{n-1}}{n!} z^n \]

\[ \hat{G}'(z) = \sum_n \frac{g_{n+1}}{n!} z^n \]

\[ \int_0^z \hat{G}(w)dw = \sum_n \frac{g_{n-1}}{n!} z^n \]

\[ \hat{F}(z) \cdot \hat{G}(z) = \sum_n \frac{1}{n!} \left( \sum_k \binom{n}{k} f_k g_{n-k} \right) z^n \]
Let \( \hat{F}(z) \) and \( \hat{G}(z) \) be the exponential generating functions of \( \langle f_n \rangle \) and \( \langle g_n \rangle \).

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\[
\begin{align*}
\alpha \hat{F}(z) + \beta \hat{G}(z) &= \sum_n \left( \frac{\alpha f_n + \beta g_n}{n!} \right) z^n \\
\hat{G}(cz) &= \sum_n \frac{c^n g_n}{n!} z^n \\
z \hat{G}(z) &= \sum_n \frac{n g_{n-1}}{n!} z^n \\
\hat{G}'(z) &= \sum_n \frac{g_{n+1}}{n!} z^n \\
\int_0^z \hat{G}(w)dw &= \sum_n \frac{g_{n-1}}{n!} z^n \\
\hat{F}(z) \cdot \hat{G}(z) &= \sum_n \frac{1}{n!} \left( \sum_k \binom{n}{k} f_k g_{n-k} \right) z^n
\end{align*}
\]
Let $\hat{F}(z)$ and $\hat{G}(z)$ be the exponential generating functions of $\langle f_n \rangle$ and $\langle g_n \rangle$.

As usual, we put $f_n = g_n = 0$ for every $n < 0$, and undefined $\cdot 0 = 0$.

- $\alpha \hat{F}(z) + \beta \hat{G}(z) = \sum_n \left( \frac{\alpha f_n + \beta g_n}{n!} \right) z^n$
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Let \( \hat{F}(z) \) and \( \hat{G}(z) \) be the exponential generating functions of \( \langle f_n \rangle \) and \( \langle g_n \rangle \).

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Let $\hat{F}(z)$ and $\hat{G}(z)$ be the exponential generating functions of $\langle f_n \rangle$ and $\langle g_n \rangle$.

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Binomial convolution

**Definition**

The binomial convolution of the sequences $\langle f_n \rangle$ and $\langle g_n \rangle$ is the sequence $\langle h_n \rangle$ defined by:

$$h_n = \sum_k \binom{n}{k} f_k g_{n-k}$$
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\[
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\]

**Examples**

- \( \langle (a + b)^n \rangle \) is the binomial convolution of \( \langle a^n \rangle \) and \( \langle b^n \rangle \).
- If \( \hat{F}(z) \) is the egf of \( \langle f_n \rangle \) and \( \hat{G}(z) \) is the egf of \( \langle g_n \rangle \), then \( \hat{H}(z) = \hat{F}(z) \cdot \hat{G}(z) \) is the egf of \( \langle h_n \rangle \), because then:

\[
\frac{h_n}{n!} = \sum_k \frac{f_k}{k!} \frac{g_{n-k}}{(n-k)!}
\]
Recall that the Bernoulli numbers are defined by the recurrence:

\[
\sum_{k=0}^{m} \binom{m+1}{k} B_k = [m = 0] \quad \forall m \geq 0,
\]

which is equivalent to:

\[
\sum_{n} \binom{n}{k} B_k = B_n + [n = 1] \quad \forall n \geq 0.
\]

The left-hand side is a binomial convolution with the constant sequence 1. Then the egf \(\hat{B}(z)\) of the Bernoulli numbers satisfies

\[
\hat{B}(z) \cdot e^z = \hat{B}(z) + z:
\]

which yields

\[
\hat{B}(z) = \frac{z}{e^z - 1}.
\]
Bernoulli numbers and exponential generating functions

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which is equivalent to:

\[ \sum_{n} \binom{n}{k} B_k = B_n + [n = 1] \quad \forall n \geq 0. \]

To make a comparison:

\[ \sum_{n \geq 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1} \quad \text{but} \quad \sum_{n \geq 0} B_n^+ z^n = \frac{1}{z} \frac{d^2}{dz^2} \ln \int_{0}^{\infty} t^{z-1} e^{-t} dt \]

where \( B_n^+ = B_n \cdot [B_n \geq 0]. \)